Random burning of the Euclidean torus

Guillaume Blanc

École Polytechnique Fédérale de Lausanne

GeoSto26, Rouen

Figure: Joint work with Alice Contat (CNRS Researcher at Université Sorbonne Paris-Nord).

The burning number

Let $G = (V, E)$ be a finite graph. The **burning number** of G is the minimal number of steps needed to burn all of its vertices, with the following procedure: at each step $i \in \mathbb{N}^*$, one can choose a point $x_i \in V$ to set on fire, and the fire propagates constantly at unit speed along the edges of the graph.

The burning number

Let $G = (V, E)$ be a finite graph. The **burning number** of G is the minimal number of steps needed to burn all of its vertices, with the following procedure: at each step $i \in \mathbb{N}^*$, one can choose a point $x_i \in V$ to set on fire, and the fire propagates constantly at unit speed along the edges of the graph.

Formally, we have

$$b(G) := \min \left\{ k \in \mathbb{N} : \exists x_1, \dots, x_k \in V \text{ s.t. } \bigcup_{i=1}^{k} \overline{B}(x_i, k-i) = V \right\}.$$

The burning number

Let $G = (V, E)$ be a finite graph. The **burning number** of G is the minimal number of steps needed to burn all of its vertices, with the following procedure: at each step $i \in \mathbb{N}^*$, one can choose a point $x_i \in V$ to set on fire, and the fire propagates constantly at unit speed along the edges of the graph.

Formally, we have

$$b(G) := \min \left\{ k \in \mathbb{N} : \exists x_1, \dots, x_k \in V \text{ s.t. } \bigcup_{i=1}^{k} \overline{B}(x_i, k-i) = V \right\}.$$

Originally introduced by Alon (1992), and reintroduced more recently under that name by Bonato, Janssen and Roshanbin (2016).

The burning number conjecture

For the path P_n on n vertices, we have $b(P_n) = \lceil \sqrt{n} \rceil$.

The burning number conjecture

For the path P_n on n vertices, we have $b(P_n) = \lceil \sqrt{n} \rceil$.

Proof.

Lower bound: Set $k = b(P_n)$ for short, and fix $x_1, \ldots, x_k \in [\![1, n]\!]$ such that $\bigcup_{i=1}^{k} \overline{B}(x_i, k-i) = [\![1, n]\!]$. We have

$$n \leq \sum_{i=1}^{k} \#\overline{B}(x_i, k-i) \leq \sum_{i=1}^{k} (2(k-i)+1) = \sum_{j=0}^{k-1} (2j+1) = k^2,$$

hence $k \geq \lceil \sqrt{n} \rceil$.

□

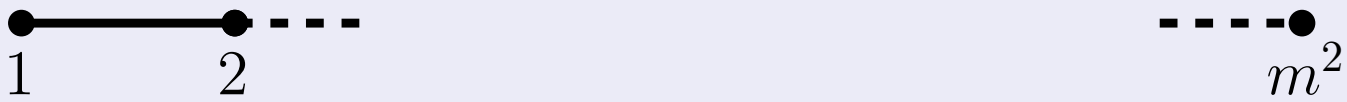
The burning number conjecture

For the path P_n on n vertices, we have $b(P_n) = \lceil \sqrt{n} \rceil$.

Proof.

Lower bound: $b(P_n) \geq \lceil \sqrt{n} \rceil$.

Upper bound: By monotonicity, it suffices to show that $b(P_{m^2}) \leq m$.



□

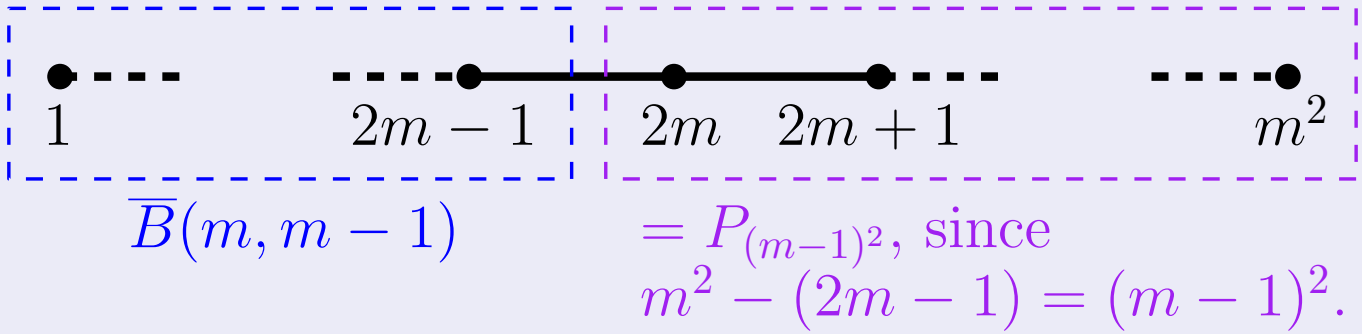
The burning number conjecture

For the path P_n on n vertices, we have $b(P_n) = \lceil \sqrt{n} \rceil$.

Proof.

Lower bound: $b(P_n) \geq \lceil \sqrt{n} \rceil$.

Upper bound: By monotonicity, it suffices to show that $b(P_{m^2}) \leq m$.



□

The burning number conjecture

For the path P_n on n vertices, we have $b(P_n) = \lceil \sqrt{n} \rceil$.

Conjecture (Bonato, Janssen and Roshanbin (2016))

For any connected graph G on n vertices, we have $b(G) \leq \lceil \sqrt{n} \rceil$.

The burning number conjecture

For the path P_n on n vertices, we have $b(P_n) = \lceil \sqrt{n} \rceil$.

Conjecture (Bonato, Janssen and Roshanbin (2016))

For any connected graph G on n vertices, we have $b(G) \leq \lceil \sqrt{n} \rceil$.

This problem has received a lot of attention from the graph-theory/combinatorics community, culminating in:

Theorem (Norin, Turcotte (2024))

The burning number conjecture holds asymptotically: as $n \to \infty$, we have

$$\max\{b(G)\,;\ G \text{ connected graph on } n \text{ vertices}\} \sim \sqrt{n}.$$

Random burning processes: coupon collector

Burning number: *minimal* number steps needed to burn the graph.

Random burning: number of steps needed to burn the graph *at random*.

Random burning processes: coupon collector

Burning number: *minimal* number steps needed to burn the graph.

Random burning: number of steps needed to burn the graph *at random*.

- **Coupon collector:** Let $X_1, X_2, \ldots$ be independent random variables with uniform distribution on V.

  We define the **coupon collector random burning number** of G as:

$$\tau_{\mathrm{cc}}(G) := \inf \left\{ k \in \mathbb{N} : \bigcup_{i=1}^{k} \overline{B}(X_i, k-i) = V \right\}.$$

Random burning processes: rejection sampling

- **Rejection sampling:** Let $(Y_1, Y_2, \ldots)$ be a sequence of random variables with values in V such that for each $k \in \mathbb{N}$, the law of Y_{k+1} conditionally on $Y_1, \ldots, Y_k$ is the uniform distribution on the complement of

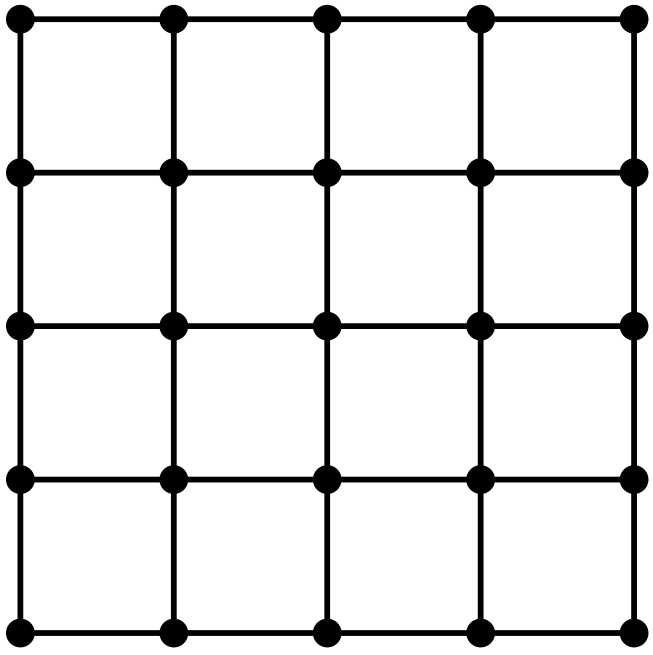
$$\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i),$$

provided that $\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) \subsetneq V$.

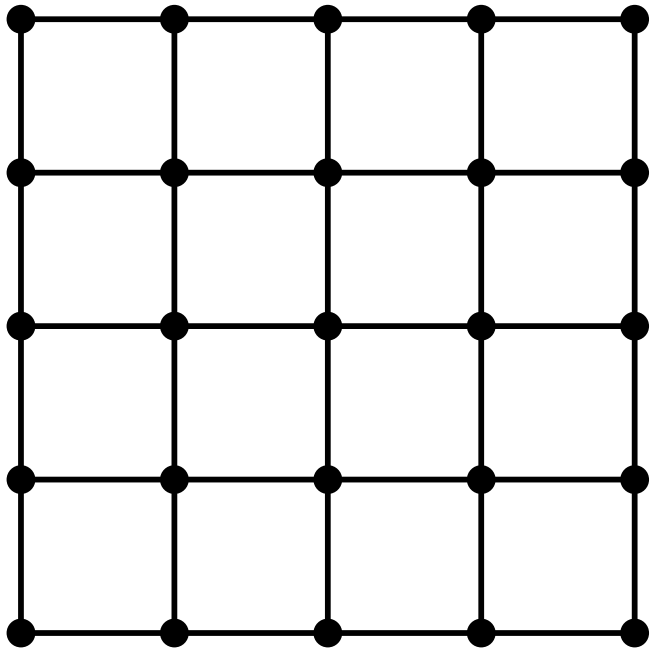
We define the **rejection sampling random burning number** of G as:

$$\tau_{\mathsf{rs}}(G) := \inf \left\{ k \in \mathbb{N} : \bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) = V \right\}.$$

Random burning processes

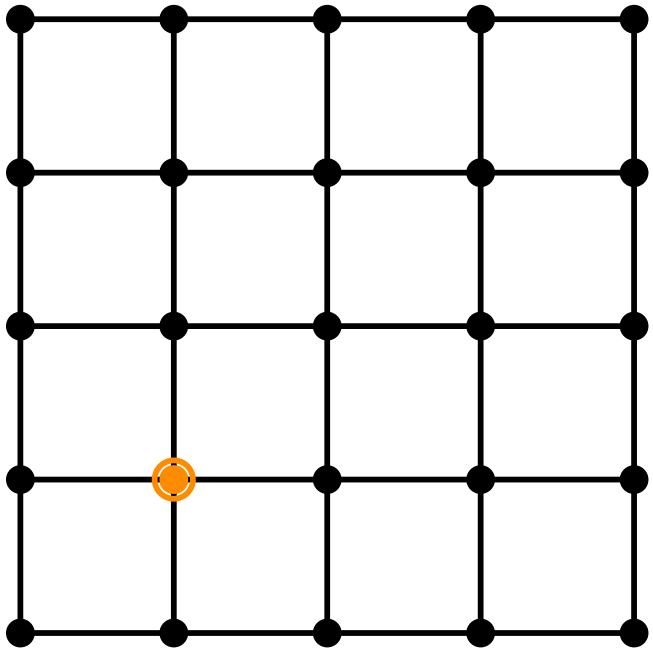


Coupon collector
Step $k = 0$

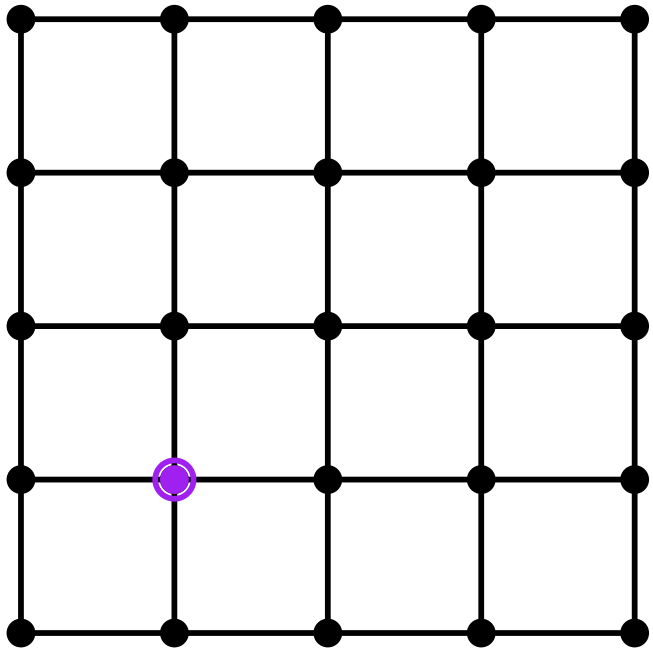


Rejection sampling
Step $k = 0$

Random burning processes

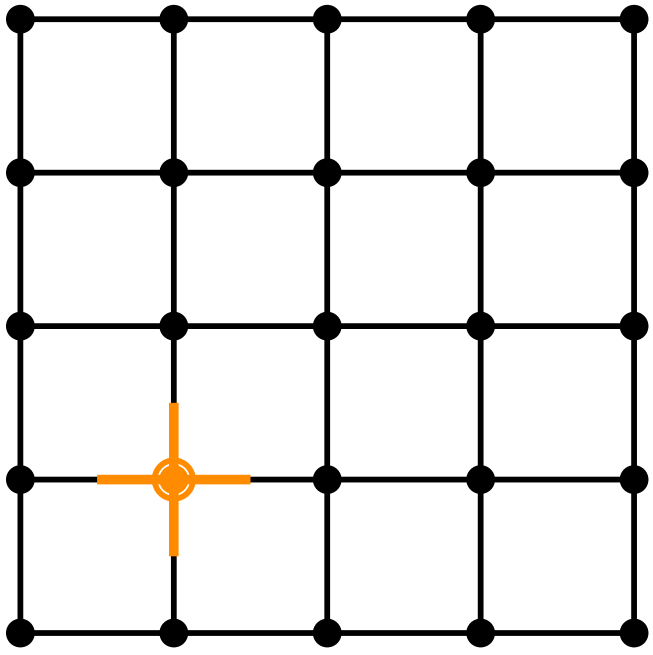


Coupon collector
Step $k = 1$

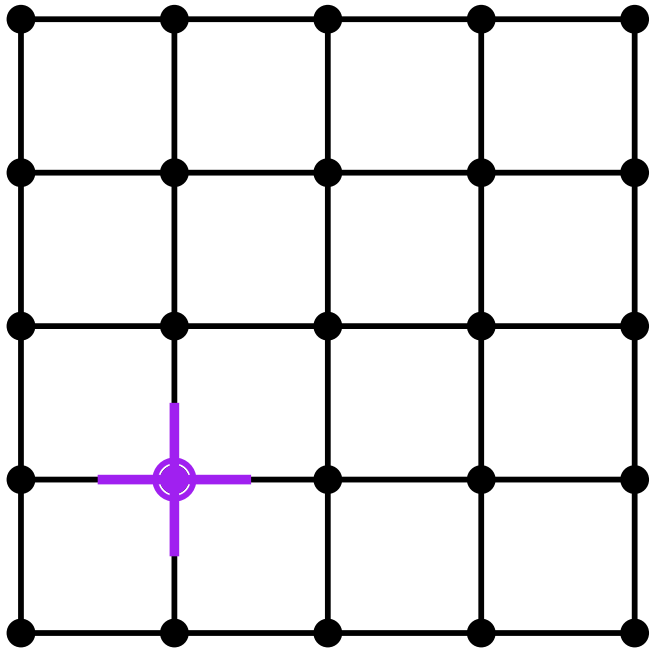


Rejection sampling
Step $k = 1$

Random burning processes

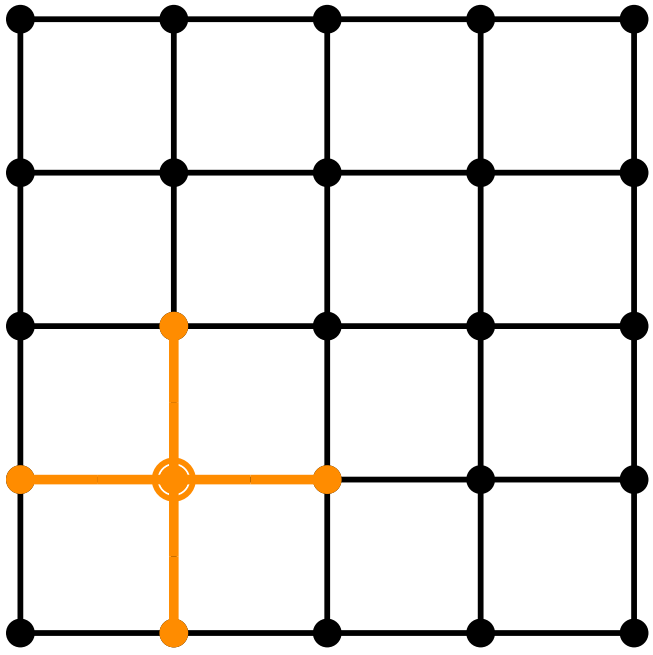


Coupon collector
Step $k = 1 + 1/2$

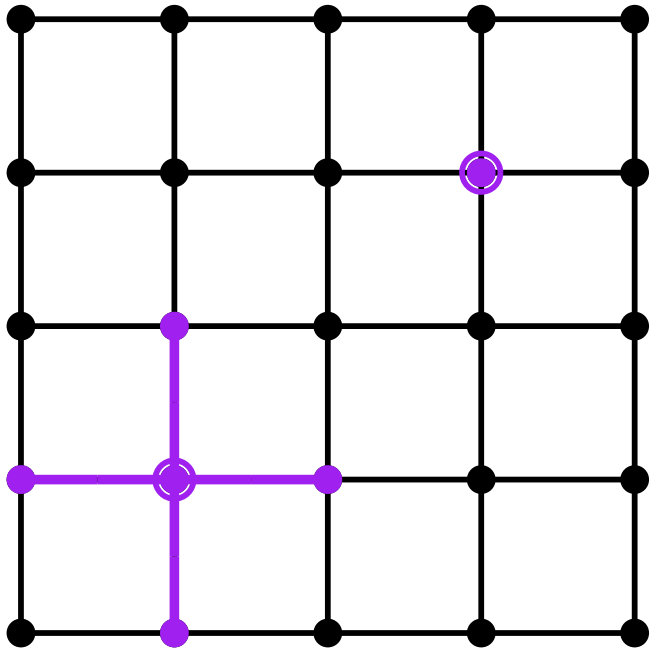


Rejection sampling
Step $k = 1 + 1/2$

Random burning processes

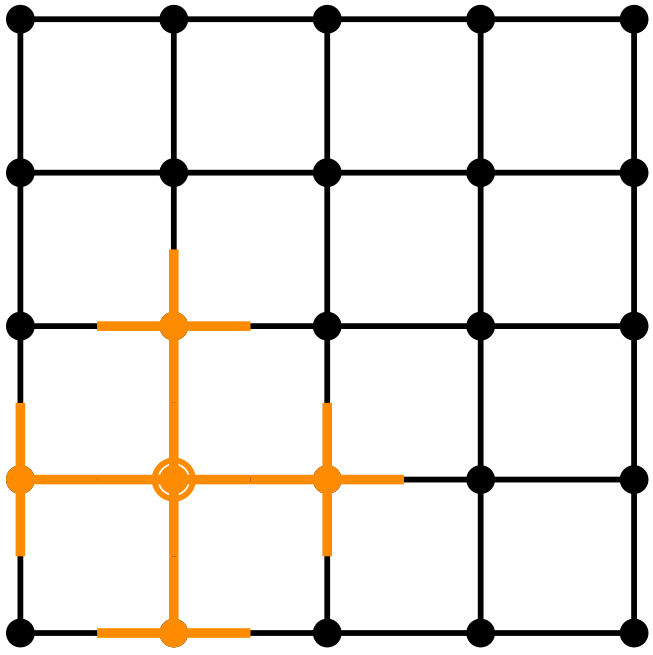


Coupon collector
Step $k = 2$

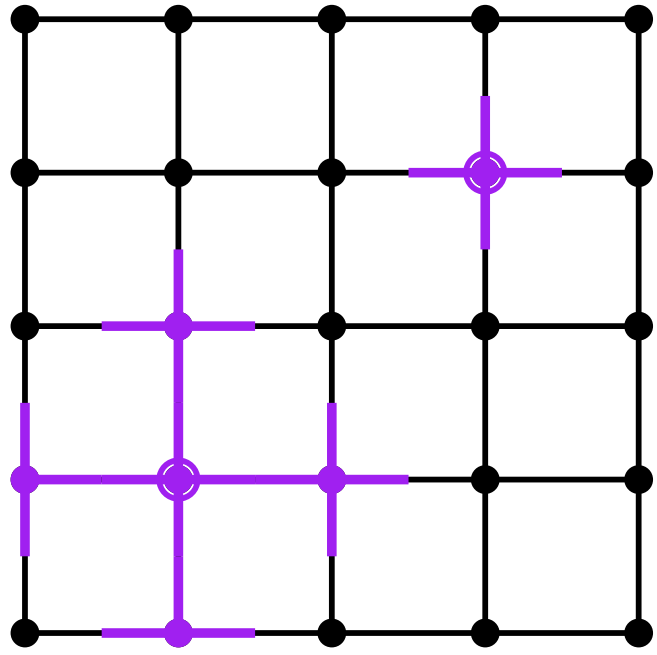


Rejection sampling
Step $k = 2$

Random burning processes

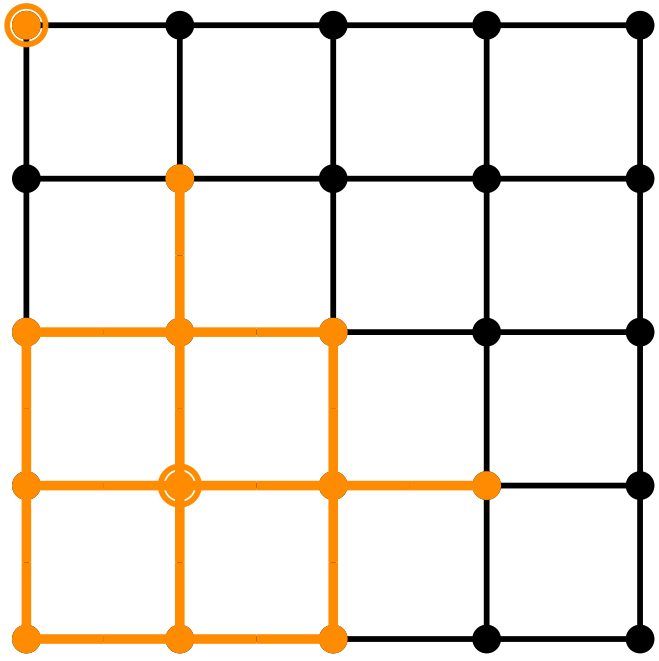


Coupon collector
Step $k = 2 + 1/2$

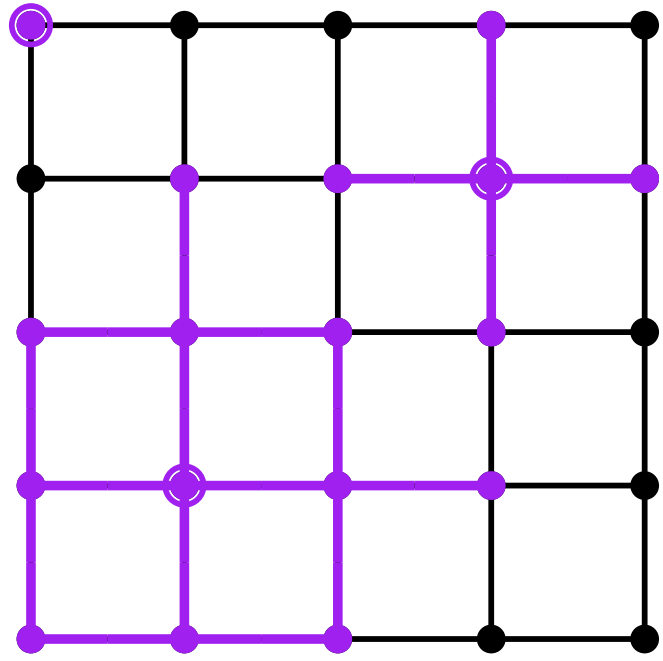


Rejection sampling
Step $k = 2 + 1/2$

Random burning processes

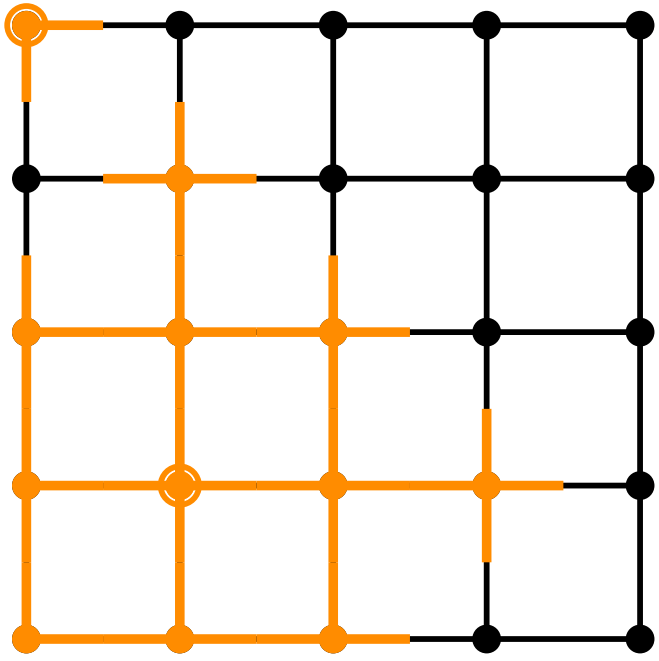


Coupon collector
Step $k = 3$

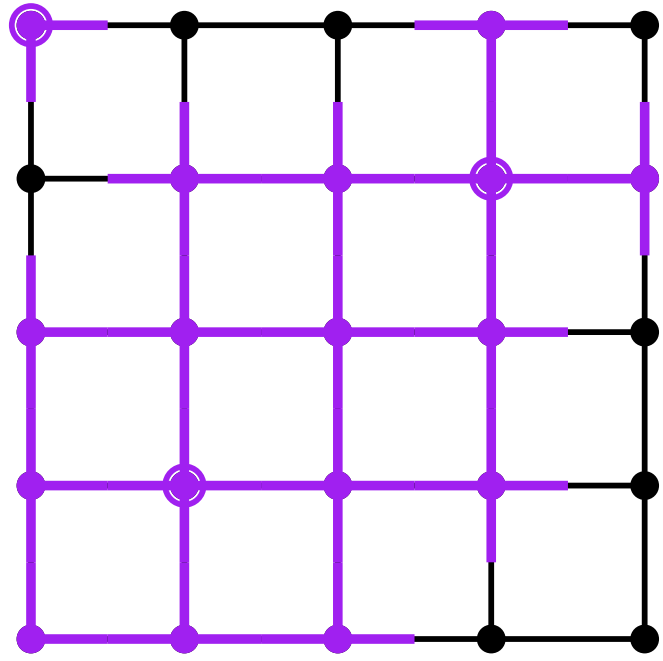


Rejection sampling
Step $k = 3$

Random burning processes

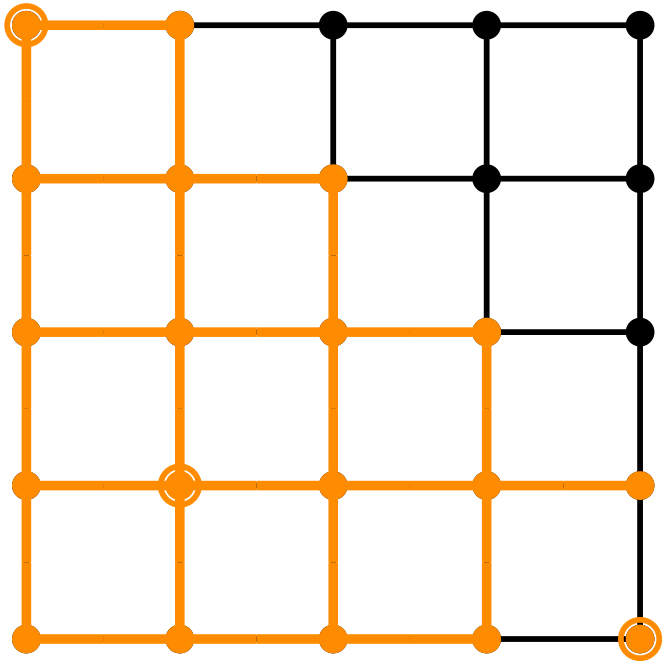


Coupon collector
Step $k = 3 + 1/2$

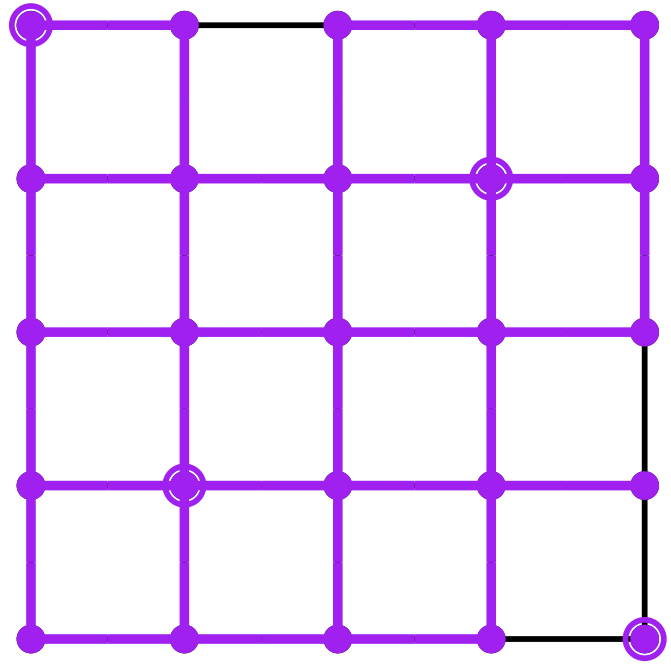


Rejection sampling
Step $k = 3 + 1/2$

Random burning processes

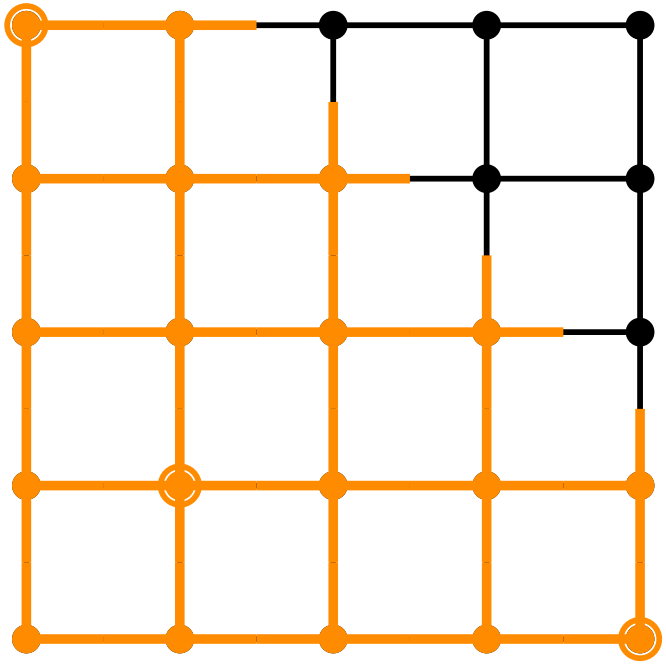


Coupon collector
Step $k = 4$

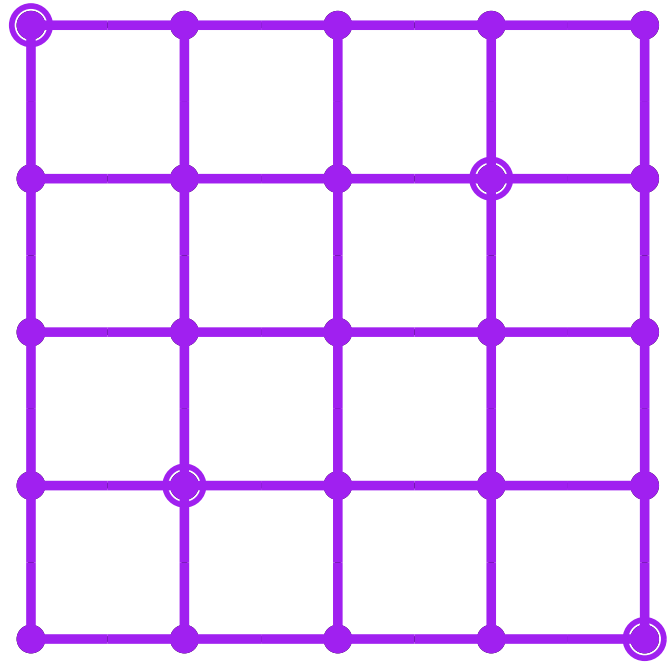


Rejection sampling
Step $k = 4$

Random burning processes

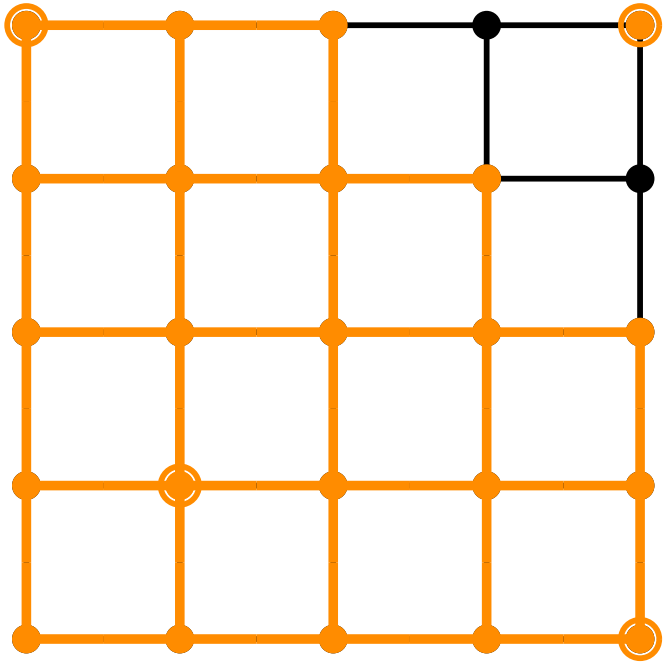


Coupon collector
Step $k = 4 + 1/2$

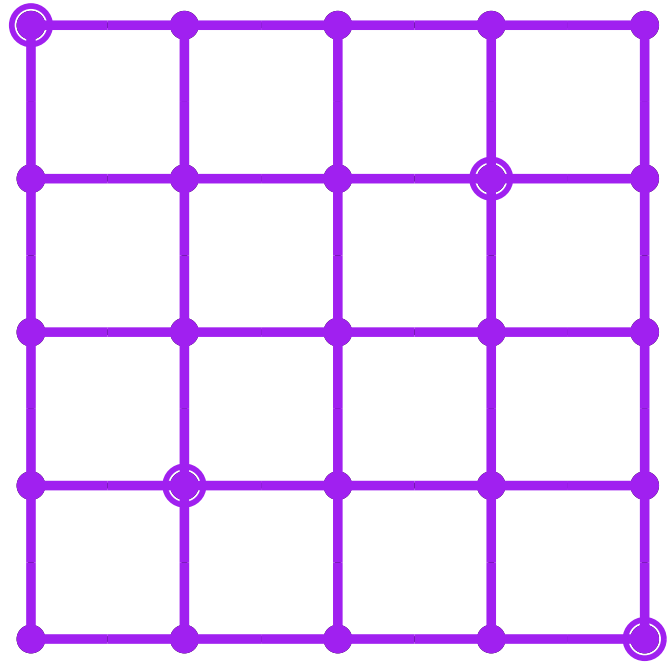


Rejection sampling
$\tau_{\mathsf{rs}}(G) = 4$

Random burning processes

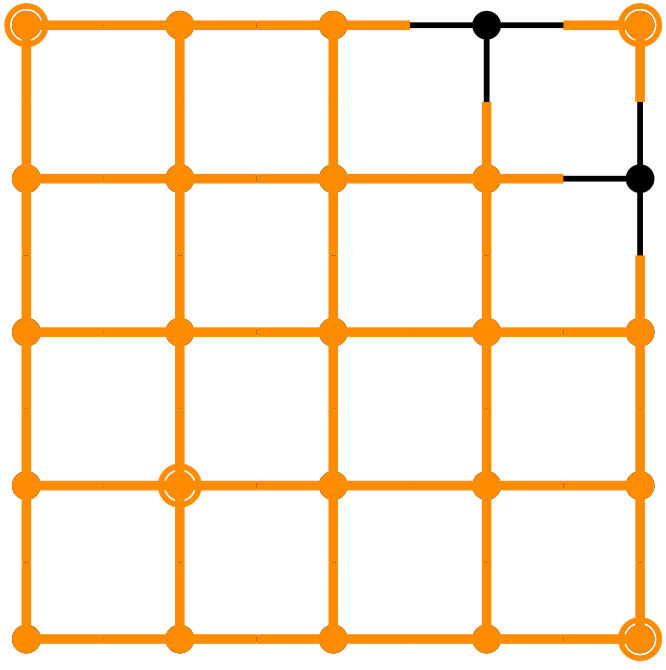


Coupon collector
Step $k = 5$

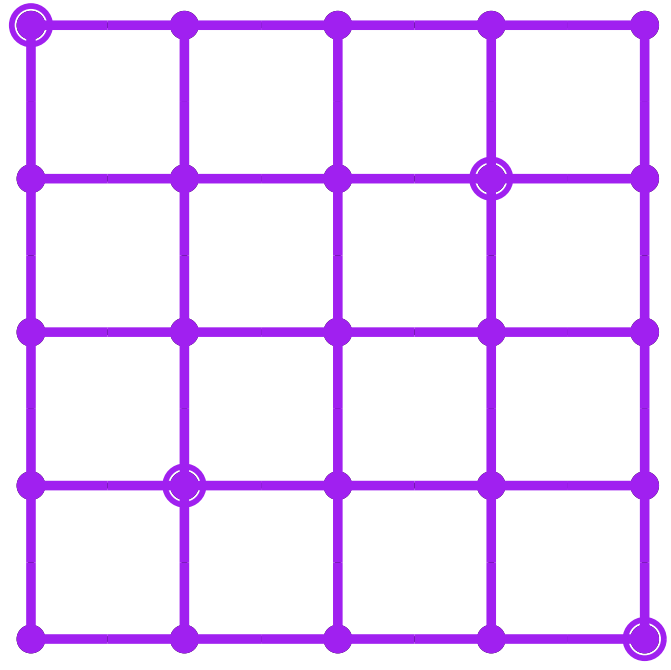


Rejection sampling
$\tau_{\mathsf{rs}}(G) = 4$

Random burning processes

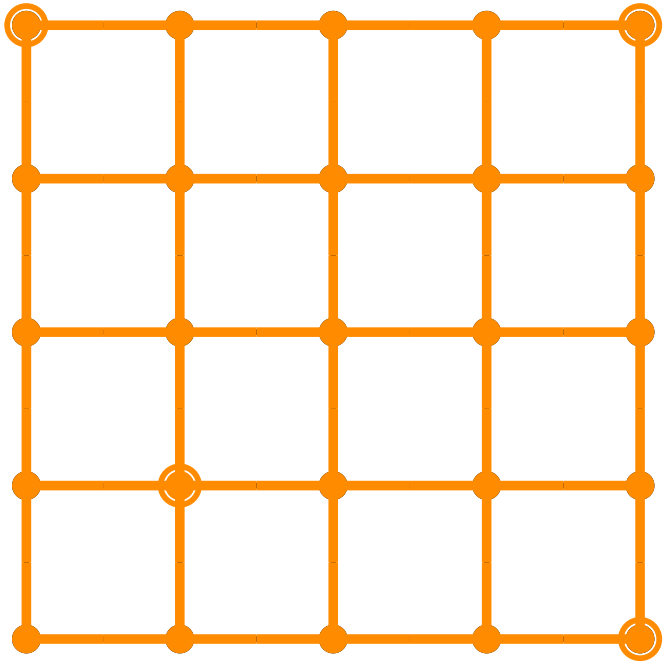


Coupon collector
Step $k = 5 + 1/2$

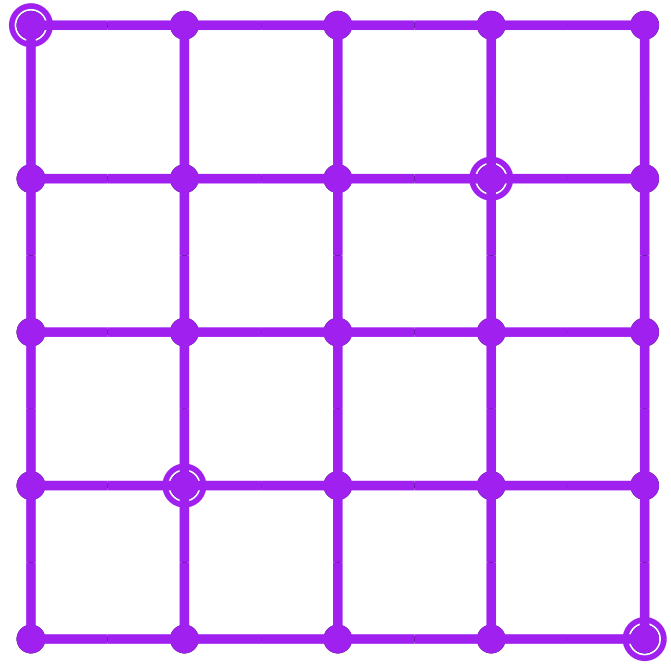


Rejection sampling
$\tau_{\mathsf{rs}}(G) = 4$

Random burning processes

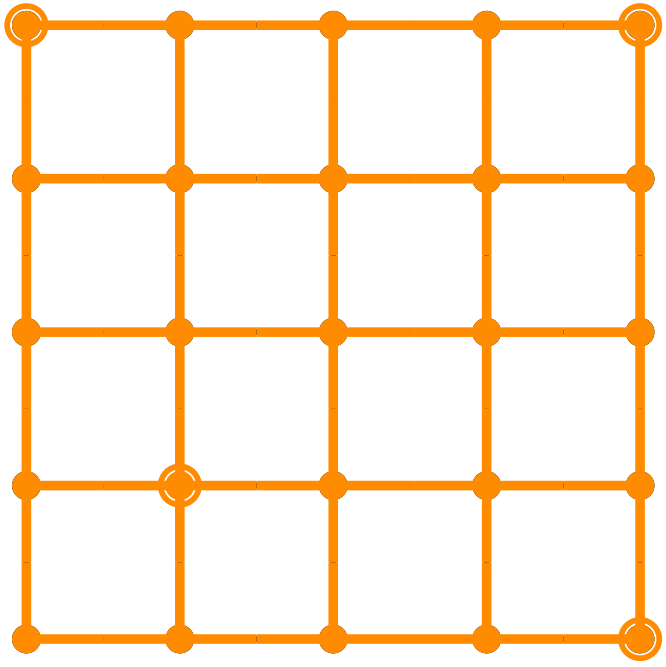


Coupon collector
Step $k = 6$



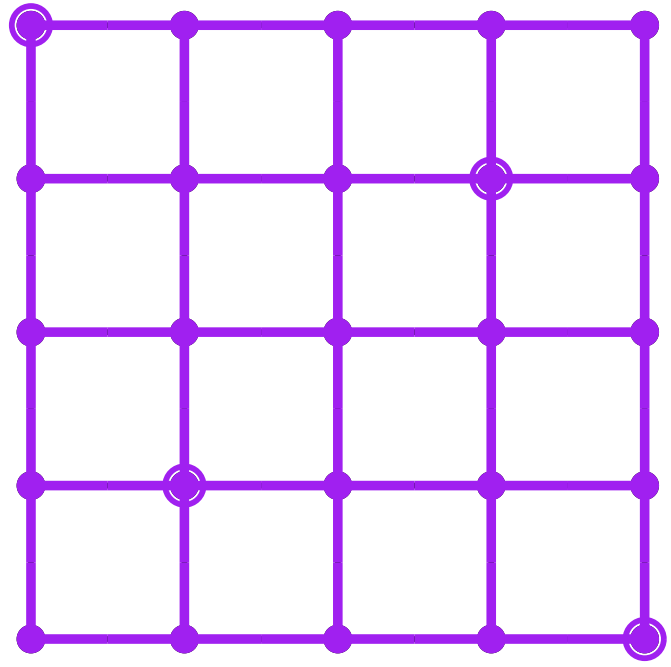
Rejection sampling
$\tau_{\mathsf{rs}}(G) = 4$

Random burning processes



Coupon collector

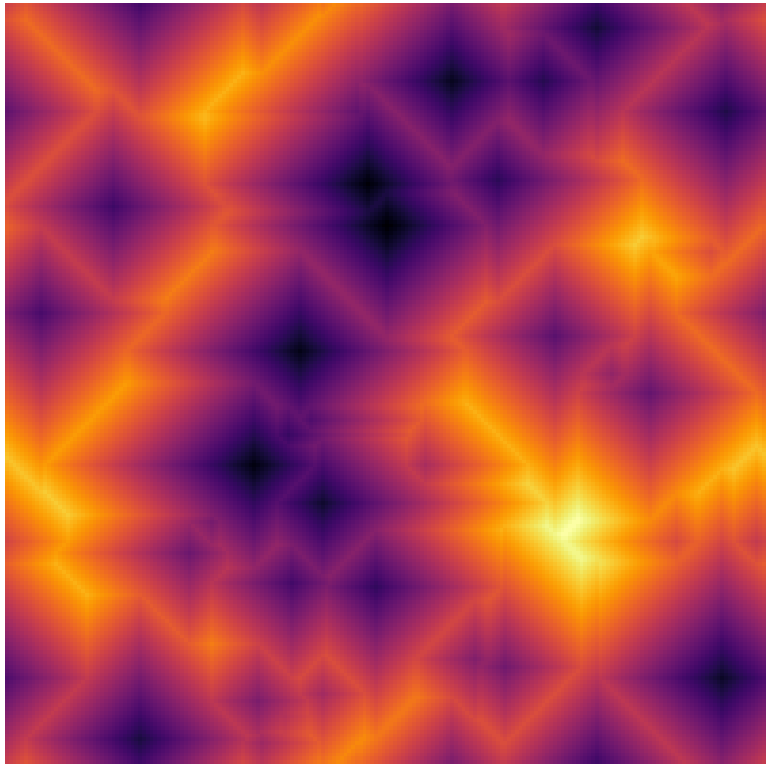
$\tau_{\mathsf{cc}}(G) = 6$



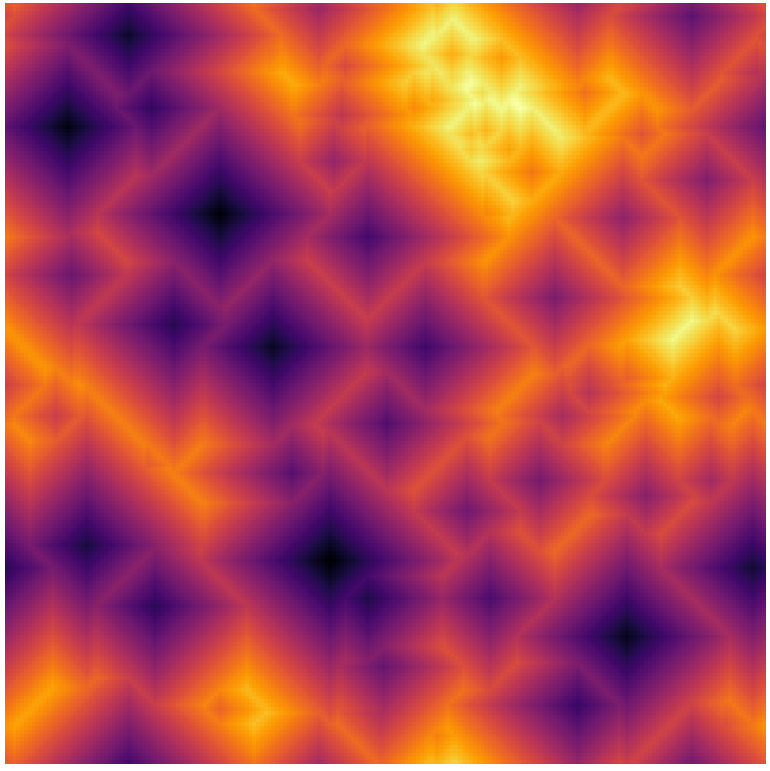
Rejection sampling

$\tau_{\mathsf{rs}}(G) = 4$

Random burning of $\mathbb{T}^2_{200}$

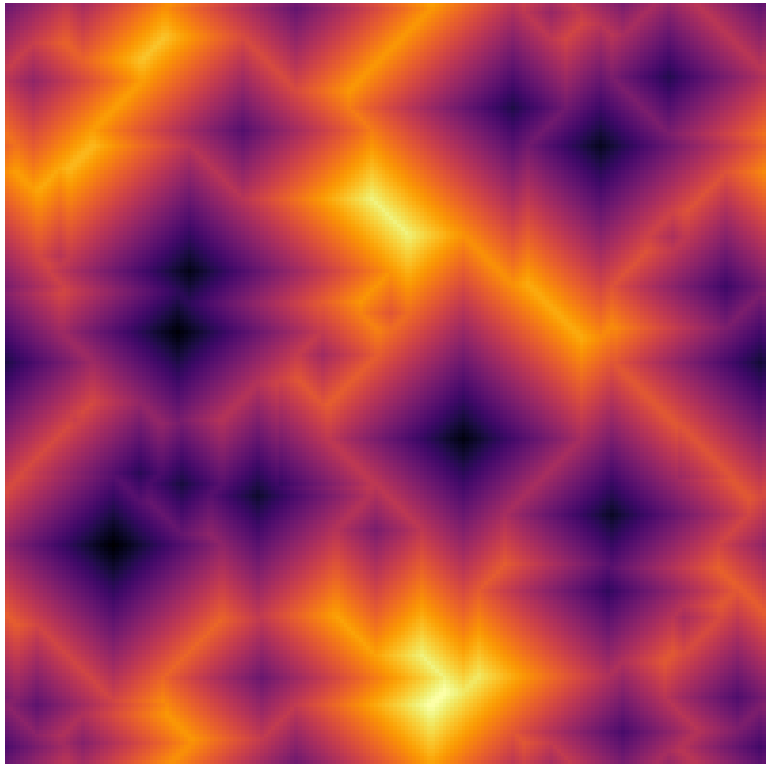


Coupon collector

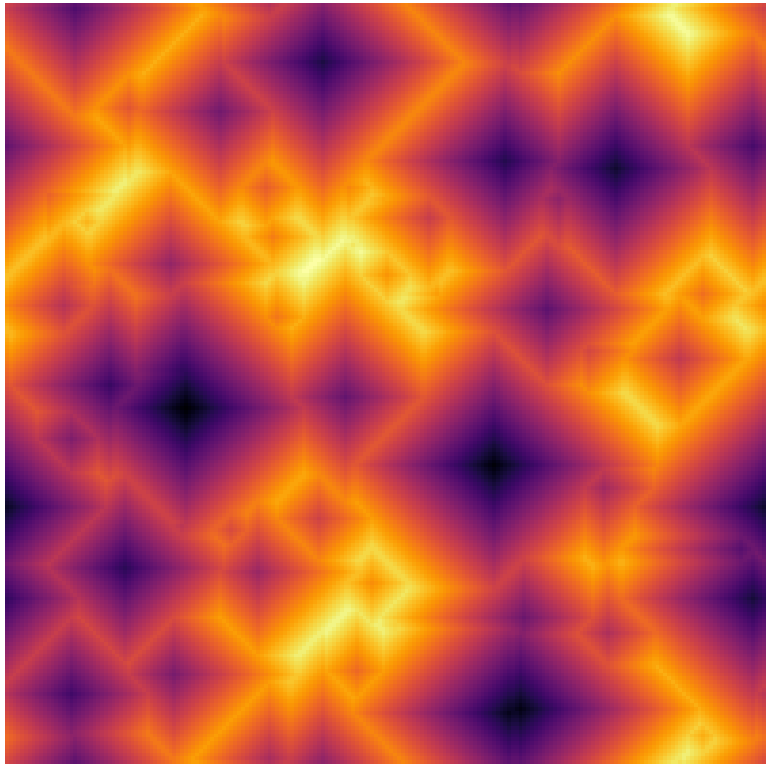


Rejection sampling

Random burning of $\mathbb{T}^2_{200}$



Coupon collector



Rejection sampling

Coupon collector random burning of P_n

Let P_n be the path on n vertices. Recall that $b(P_n) = \lceil \sqrt{n} \rceil$.

Coupon collector random burning of P_n

Let P_n be the path on n vertices. Recall that $b(P_n) = \lceil \sqrt{n} \rceil$. Recall that

$$\tau_{\mathrm{cc}}(P_n) = \inf \left\{ k \in \mathbb{N} : \bigcup_{i=1}^{k} \overline{B}(X_i, k-i) = [\![1, n]\!] \right\},$$

where $X_1, X_2, \ldots$ are independent random variables with uniform distribution on $[\![1, n]\!]$.

Coupon collector random burning of P_n

Let P_n be the path on n vertices. Recall that $b(P_n) = \lceil \sqrt{n} \rceil$. Recall that

$$\tau_{\mathrm{cc}}(P_n) = \inf \left\{ k \in \mathbb{N} : \bigcup_{i=1}^{k} \overline{B}(X_i, k-i) = [\![1, n]\!] \right\},$$

where $X_1, X_2, \ldots$ are independent random variables with uniform distribution on $[\![1, n]\!]$.

Theorem (Mitsche, Prałat and Roshanbin (2017))

As $n \to \infty$, we have

$$\frac{\tau_{\mathrm{cc}}(P_n)}{\sqrt{n \log(n)/2}} \longrightarrow 1 \quad \textit{in probability.}$$

Geometric coupon collector phenomenon.

Rejection sampling random burning of P_n

Recall that

$$\tau_{\mathsf{rs}}(P_n) = \inf\left\{ k \in \mathbb{N} : \bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) = [\![1, n]\!] \right\},$$

where for each $k \in \mathbb{N}$, the law of Y_{k+1} conditionally on $Y_1, \ldots, Y_k$ is the uniform distribution on the complement of

$$\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i),$$

provided that $\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) \subsetneq [\![1, n]\!]$.

Rejection sampling random burning of P_n

Theorem (Mitsche, Prałat and Roshanbin (2017))

There exists a constant $C \in]1, \infty[$ *such that*

$$\mathbb{P}\left(1 \leq \frac{\tau_{\mathsf{rs}}(P_n)}{\sqrt{n}} \leq C\right) \longrightarrow 1 \quad \text{as } n \to \infty.$$

Achieves the same order as $b(P_n) = \lceil \sqrt{n} \rceil$.

Random burning of the discrete Euclidean torus

Let $\mathbb{T}_n^d = \mathbb{Z}^d / n\mathbb{Z}^d$ be the discrete d-dimensional Euclidean torus with side-length n.

Goal: Study the asymptotic behaviour of $\tau_{\mathsf{cc}}\left(\mathbb{T}_n^d\right)$ and $\tau_{\mathsf{rs}}\left(\mathbb{T}_n^d\right)$ as $n \to \infty$.

Random burning of the discrete Euclidean torus

Let $\mathbb{T}_n^d = \mathbb{Z}^d / n\mathbb{Z}^d$ be the discrete d-dimensional Euclidean torus with side-length n.

Goal: Study the asymptotic behaviour of $\tau_{\mathsf{cc}}\left(\mathbb{T}_n^d\right)$ and $\tau_{\mathsf{rs}}\left(\mathbb{T}_n^d\right)$ as $n \to \infty$.

The results of Mitsche, Prałat and Roshanbin for P_n also apply for $\mathbb{T}_n^1$: we have

$$\frac{\tau_{\mathsf{cc}}\left(\mathbb{T}_n^1\right)}{\sqrt{n\log(n)/2}} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty,$$

and there exists a constant $C \in]1, \infty[$ such that

$$\mathbb{P}\left(1 \leq \frac{\tau_{\mathsf{rs}}\left(\mathbb{T}_n^1\right)}{\sqrt{n}} \leq C\right) \longrightarrow 1 \quad \text{as } n \to \infty.$$

Burning number of $\mathbb{T}_n^d$

As for P_n, we have $b\left(\mathbb{T}_n^1\right) = \left\lceil \sqrt{n} \right\rceil$.

Burning number of $\mathbb{T}_n^d$

As for P_n, we have $b\left(\mathbb{T}_n^1\right) = \left\lceil \sqrt{n} \right\rceil$.

Theorem (Mitsche, Prałat and Roshanbin (2017))

$$b\left(\mathbb{T}_n^2\right) \sim \left(\frac{3}{2}\right)^{1/3} \cdot n^{2/3} \quad \text{as } n \to \infty.$$

Burning number of $\mathbb{T}_n^d$

As for P_n, we have $b\left(\mathbb{T}_n^1\right) = \left\lceil \sqrt{n} \right\rceil$.

Theorem (Mitsche, Prałat and Roshanbin (2017))

$$b\left(\mathbb{T}_n^2\right) \sim \left(\frac{3}{2}\right)^{1/3} \cdot n^{2/3} \quad \text{as } n \to \infty.$$

Proposition

As $n \to \infty$, we have

$$(\alpha_d + o(1)) \cdot n^{d/(d+1)} \leq b\left(\mathbb{T}_n^d\right) \leq (\gamma_d + o(1)) \cdot n^{d/(d+1)},$$

where

$$\alpha_d = \left(\frac{(d+1)!}{2^d}\right)^{1/(d+1)}.$$

Coupon collector random burning of $\mathbb{T}_n^d$

Recall that

$$\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) = \inf\left\{k \in \mathbb{N} : \bigcup_{i=1}^{k} \overline{B}(X_i, k-i) = \mathbb{T}_n^d\right\},$$

where $X_1, X_2, \ldots$ are independent random variables with uniform distribution on $\mathbb{T}_n^d$.

Theorem (with A. Contat (2025+))

As $n \to \infty$, we have

$$\frac{\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right)}{\beta_d \cdot n^{d/(d+1)} \cdot \log(n)^{1/(d+1)}} \longrightarrow 1 \quad \textit{in probability,}$$

where $\beta_d = \left(\frac{d! \cdot d}{2^d}\right)^{1/(d+1)}$.

Geometric coupon collector phenomenon.

Rejection sampling random burning of $\mathbb{T}_n^d$

Recall that

$$\tau_{\mathsf{rs}}\left(\mathbb{T}_n^d\right) = \inf\left\{k \in \mathbb{N} : \bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) = \mathbb{T}_n^d\right\},$$

where for each $k \in \mathbb{N}$, the law of Y_{k+1} conditionally on $Y_1, \ldots, Y_k$ is the uniform distribution on the complement of

$$\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i),$$

provided that $\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) \subsetneq \mathbb{T}_n^d$.

Rejection sampling random burning of $\mathbb{T}_n^d$

Theorem (with A. Contat (2025+))

As $n \to \infty$, we have

$$\frac{\tau_{\mathsf{rs}}\left(\mathbb{T}_n^d\right)}{n^{d/(d+1)}} \longrightarrow T \quad \textit{in probability,}$$

where $T = T(d) \in (0, \infty)$ is the explosion time of the maximal solution $y : [0, T[\to \mathbb{R}$ to the generalised Blasius equation:

$$\begin{cases} y^{(d+1)}(t) = 2^d \cdot y(t) \cdot y^{(d)}(t), \quad t \in \mathbb{R}_+, \\ y(0), \ldots, y^{(d-1)}(0) = 0 \text{ and } y^{(d)}(0) = 1. \end{cases}$$

Achieves the same order as $b\left(\mathbb{T}_n^d\right) \underset{n\to\infty}{\asymp} n^{d/(d+1)}$.

Rejection sampling random burning of $\mathbb{T}_n^d$

Set

$$\mathcal{B}_n^{\mathrm{rs}}(k) := \bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) \quad \text{for all } k \in \mathbb{N}.$$

Theorem (with A. Contat (2025+))

As $n \to \infty$, we have

$$\sup_{t \in [0,T[} \left| \frac{\#\left(\mathbb{T}_n^d \setminus \mathcal{B}_n^{\mathrm{rs}}\left(\lfloor t \cdot n^{d/(d+1)} \rfloor\right)\right)}{n^d} - \frac{1}{y^{(d)}(t)} \right| \longrightarrow 0 \quad \textit{in probability,}$$

where $y : [0, T[\to \mathbb{R}$ is the maximal solution of the generalised Blasius equation:

$$\begin{cases} y^{(d+1)}(t) = 2^d \cdot y(t) \cdot y^{(d)}(t), \quad t \in \mathbb{R}_+, \\ y(0), \dots, y^{(d-1)}(0) = 0 \text{ and } y^{(d)}(0) = 1. \end{cases}$$

Rejection sampling random burning of $\mathbb{T}_n^1$

For $d = 1$, the maximal solution of the generalised Blasius equation

$$\begin{cases} y''(t) = 2 \cdot y(t) \cdot y'(t), & t \in \mathbb{R}_+, \\ y(0) = 0 \text{ and } y'(0) = 1. \end{cases}$$

is explicit: we have $T = \pi/2$, and $y(t) = \tan(t)$ for all $t \in [0, \pi/2[$.

Rejection sampling random burning of $\mathbb{T}_n^1$

For $d = 1$, the maximal solution of the generalised Blasius equation

$$\begin{cases} y''(t) = 2 \cdot y(t) \cdot y'(t), & t \in \mathbb{R}_+, \\ y(0) = 0 \text{ and } y'(0) = 1. \end{cases}$$

is explicit: we have $T = \pi/2$, and $y(t) = \tan(t)$ for all $t \in [0, \pi/2[$.

In particular, we get that

$$\frac{\tau_{\mathsf{rs}}\left(\mathbb{T}_n^1\right)}{\sqrt{n}} \xrightarrow{\mathbb{P}} \frac{\pi}{2} \quad \text{as } n \to \infty,$$

and

$$\sup_{t \in [0,\pi/2[} \left| \frac{\#\left(\mathbb{T}_n^1 \backslash \mathcal{B}_n^{\mathsf{rs}}\left(\left\lfloor t \cdot \sqrt{n} \right\rfloor\right)\right)}{n} - \cos(t)^2 \right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.$$

Heuristic

Fix a function $f_n : [0, T_n[\to [1, \infty[$ to be adjusted later, and let Π_n be a Poisson process with intensity $\mathrm{d}x \otimes f_n(t)\mathrm{d}t$ on $\mathbb{T}^d \times [0, T_n[$, where $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ is the continuum d-dimensional Euclidean torus with side-length 1.

Heuristic

Fix a function $f_n : [0, T_n[\to [1, \infty[$ to be adjusted later, and let Π_n be a Poisson process with intensity $\mathrm{d}x \otimes f_n(t)\mathrm{d}t$ on $\mathbb{T}^d \times [0, T_n[$, where $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ is the continuum d-dimensional Euclidean torus with side-length 1.

For each atom (x, t) of Π_n, we set the point $x \in \mathbb{T}^d$ on fire at time $t \in [0, T_n[$, and the fire propagates constantly at speed $1/n$, with respect to the metric on $\mathbb{T}^d$ induced by the 1-norm $\| \cdot \|_1$ on $\mathbb{R}^d$ (we want to mimic the rejection sampling burning process in $\mathbb{T}^d_n$, seen from distance n).

Heuristic

Fix a function $f_n : [0, T_n[\to [1, \infty[$ to be adjusted later, and let Π_n be a Poisson process with intensity $\mathrm{d}x \otimes f_n(t)\mathrm{d}t$ on $\mathbb{T}^d \times [0, T_n[$, where $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ is the continuum d-dimensional Euclidean torus with side-length 1.

For each atom (x, t) of Π_n, we set the point $x \in \mathbb{T}^d$ on fire at time $t \in [0, T_n[$, and the fire propagates constantly at speed $1/n$, with respect to the metric on $\mathbb{T}^d$ induced by the 1-norm $\| \cdot \|_1$ on $\mathbb{R}^d$ (we want to mimic the rejection sampling burning process in $\mathbb{T}_n^d$, seen from distance n).

In this continuum burning process in $\mathbb{T}^d$, the set of elements of $\mathbb{T}^d$ that are burned by time $t \in [0, T_n[$ is

$$\mathcal{B}_n(t) = \bigcup_{\substack{(x,s)\in\Pi_n \\ s\leq t}} \overline{B}\left(x, \frac{t-s}{n}\right).$$

Heuristic

To mimic the rejection sampling burning process in $\mathbb{T}_n^d$ (seen from distance n), we want to design the function f_n so that at each time $t \in [0, T_n[$, on average, the process Π_n puts one point per unit of time in the unburned region $\mathbb{T}^d \setminus \mathcal{B}_n(t)$ (the other points do not contribute): we want

$$f_n(t) \cdot \mathbb{E}\left[\left|\mathbb{T}^d \setminus \mathcal{B}_n(t)\right|\right] = 1 \quad \text{for all } t \in [0, T_n[.$$

Heuristic

To mimic the rejection sampling burning process in $\mathbb{T}_n^d$ (seen from distance n), we want to design the function f_n so that at each time $t \in [0, T_n[$, on average, the process Π_n puts one point per unit of time in the unburned region $\mathbb{T}^d \setminus \mathcal{B}_n(t)$ (the other points do not contribute): we want

$$f_n(t) \cdot \mathbb{E}\left[\left|\mathbb{T}^d \setminus \mathcal{B}_n(t)\right|\right] = 1 \quad \text{for all } t \in [0, T_n[.$$

Now, let us calculate

$$\mathbb{E}\left[\left|\mathbb{T}^d \setminus \mathcal{B}_n(t)\right|\right] = \int_{\mathbb{T}^d} \mathbb{P}(z \notin \mathcal{B}_n(t))\mathrm{d}z.$$

Heuristic

Recall that

$$\mathcal{B}_n(t) = \bigcup_{\substack{(x,s)\in\Pi_n \\ s\le t}} \overline{B}\left(x, \frac{t-s}{n}\right).$$

We have

$$\begin{aligned}
&\mathbb{P}(z \notin \mathcal{B}_n(t)) \\
&= \mathbb{P}\left(\Pi_n\left(\left\{(x,s)\in\mathbb{T}^d\times[0,t] : z\in\overline{B}\left(x,\frac{t-s}{n}\right)\right\}\right)=0\right) \\
&= \exp\left(-\int_{\mathbb{T}^d}\int_0^t 1\left(d(z,x)\le\frac{t-s}{n}\right)\cdot f_n(s)\mathrm{d}s\mathrm{d}x\right) \\
&= \exp\left(-\int_0^t\left|\overline{B}\left(z,\frac{t-s}{n}\right)\right|\cdot f_n(s)\mathrm{d}s\right) \\
&= \exp\left(-\int_0^t\frac{2^d}{d!}\cdot\left(\frac{t-s}{n}\right)^d\cdot f_n(s)\mathrm{d}s\right).
\end{aligned}$$

Heuristic

Thus, we get

$$\mathbb{E}\left[\left|\mathbb{T}^d \setminus \mathcal{B}_n(t)\right|\right] = \exp\left(-\int_0^t \frac{2^d}{d!} \cdot \left(\frac{t-s}{n}\right)^d \cdot f_n(s)\mathrm{d}s\right).$$

Heuristic

Thus, we get

$$\mathbb{E}\left[\left|\mathbb{T}^d \setminus \mathcal{B}_n(t)\right|\right] = \exp\left(-\int_0^t \frac{2^d}{d!} \cdot \left(\frac{t-s}{n}\right)^d \cdot f_n(s)\mathrm{d}s\right).$$

For our purpose, we want

$$f_n(t) \cdot \exp\left(-\int_0^t \frac{2^d}{d!} \cdot \left(\frac{t-s}{n}\right)^d \cdot f_n(s)\mathrm{d}s\right) = 1 \quad \text{for all } t \in [0, T_n[.$$

Heuristic

Thus, we get

$$\mathbb{E}\left[\left|\mathbb{T}^d \setminus \mathcal{B}_n(t)\right|\right] = \exp\left(-\int_0^t \frac{2^d}{d!} \cdot \left(\frac{t-s}{n}\right)^d \cdot f_n(s)\mathrm{d}s\right).$$

For our purpose, we want

$$f_n(t) \cdot \exp\left(-\int_0^t \frac{2^d}{d!} \cdot \left(\frac{t-s}{n}\right)^d \cdot f_n(s)\mathrm{d}s\right) = 1 \quad \text{for all } t \in [0, T_n[.$$

This entails (...) that $f_n(t) = y^{(d)}\left(t \cdot n^{-d/(d+1)}\right)$ for all $t \in [0, T_n[$, where $y : [0, T[\to \mathbb{R}$ is the maximal solution of the generalised Blasius equation:

$$y^{(d)}(t) = \exp\left(\frac{2^d}{d!} \cdot \int_0^t (t-s)^d \cdot y^{(d)}(s)\mathrm{d}s\right) \quad \text{for all } t \in [0, T[.$$

Heuristic

Conclusion: We find that

$$T_n = T \cdot n^{d/(d+1)},$$

and

$$\mathbb{E}\left[\left|\mathbb{T}^d \setminus \mathcal{B}_n\left(t \cdot n^{d/(d+1)}\right)\right|\right] = \frac{1}{y^{(d)}(t)} \quad \text{for all } t \in [0, T[.$$

Heuristic

Theorem (with A. Contat (2025+))

As $n \to \infty$, we have

$$\frac{\tau_{\mathsf{rs}}\left(\mathbb{T}_n^d\right)}{n^{d/(d+1)}} \longrightarrow T \quad \textit{in probability,}$$

and

$$\sup_{t\in[0,T[} \left| \frac{\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathsf{rs}}\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} - \frac{1}{y^{(d)}(t)} \right| \longrightarrow 0 \quad \textit{in probability,}$$

Blow-up rate of generalised Blasius equation

Let $y : [0, T[\to \mathbb{R}$ be the maximal solution of the generalised Blasius equation:

$$\begin{cases} y^{(d+1)}(t) = 2^d \cdot y(t) \cdot y^{(d)}(t), \quad t \in \mathbb{R}_+, \\ y(0), \dots, y^{(d-1)}(0) = 0 \text{ and } y^{(d)}(0) = 1. \end{cases}$$

Theorem (with A. Contat (2025+))

This solution blows up in finite time: we have $T < \infty$, and for each $i \in [\![0, d]\!]$, we have $y^{(i)}(t) \to \infty$ as $t \to T^-$.
Moreover, for each $i \in [\![0, d]\!]$, we have

$$y^{(i)}(t) \asymp \frac{1}{(T-t)^{i+1}} \quad \text{as } t \to T^-.$$

Questions?

Coupon collector random burning of $\mathbb{T}_n^d$

Let $X_1, X_2, \ldots$ be independent random variables with uniform distribution on $\mathbb{T}_n^d$. We set

$$\mathcal{B}_n^{\mathrm{cc}}(k) := \bigcup_{i=1}^{k} \overline{B}(X_i, k-i) \quad \text{for all } k \in \mathbb{N},$$

and we let

$$\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) = \inf\left\{k \in \mathbb{N} : \mathcal{B}_n^{\mathrm{cc}}(k) = \mathbb{T}_n^d\right\}.$$

Theorem (with A. Contat (2025+))

As $n \to \infty$, we have

$$\frac{\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right)}{\beta_d \cdot n^{d/(d+1)} \cdot \log(n)^{1/(d+1)}} \longrightarrow 1 \quad \textit{in probability,}$$

where $\beta_d = \left(\frac{d! \cdot d}{2^d}\right)^{1/(d+1)}$.

First moment argument

The proof is based on a first and second moment argument, applied to the random variables $\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}(k)\right)$. Indeed, we have

$$\left(\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) > k\right) = \left(\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k)\right) > 0\right).$$

First moment argument

The proof is based on a first and second moment argument, applied to the random variables $\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}(k)\right)$. Indeed, we have

$$\left(\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) > k\right) = \left(\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k)\right) > 0\right).$$

Now,

$$\mathbb{E}\left[\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k)\right)\right] = \sum_{x \in \mathbb{T}_n^d} \mathbb{P}\left(x \notin \mathcal{B}_n^{\mathrm{cc}}(k)\right),$$

where

$$\mathbb{P}\left(x \notin \mathcal{B}_n^{\mathrm{cc}}(k)\right) = \mathbb{P}\left(\bigcap_{i=1}^{k} \left(x \notin \overline{B}(X_i, k-i)\right)\right)$$

$$= \prod_{i=1}^{k}\left(1 - \frac{\#\overline{B}(x, k-i)}{n^d}\right) = \prod_{j=0}^{k-1}\left(1 - \frac{\#\overline{B}(0, j)}{n^d}\right).$$

First moment argument

In hindsight, we set $k_n(c) = \left\lfloor \left(c \cdot n^d \cdot \log(n)\right)^{1/(d+1)} \right\rfloor$, where $c \in \mathbb{R}_+^*$ is a parameter. We have

$$\begin{aligned}
\mathbb{P}\left(x \notin \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right) &= \prod_{j=0}^{k_n(c)-1} \left(1 - \frac{\#\overline{B}(0,j)}{n^d}\right) \\
&\approx \exp\left(-\sum_{j=0}^{k_n(c)-1} \frac{\left(2^d/d!\right) \cdot j^d}{n^d}\right) \\
&\approx \exp\left(-\frac{2^d}{(d+1)!} \cdot \frac{k_n(c)^{d+1}}{n^d}\right) \\
&\approx \exp\left(-\frac{2^d}{(d+1)!} \cdot c \cdot \log(n)\right)
\end{aligned}$$

as $n \to \infty$.

First moment argument

Thus, we have

$$\mathbb{E}\left[\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right)\right] \approx n^d \cdot \exp\left(-\frac{2^d}{(d+1)!} \cdot c \cdot \log(n)\right)$$

as $n \to \infty$, hence

$$\begin{aligned}
\mathbb{P}\left(\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) > k_n(c)\right) &= \mathbb{P}\left(\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right) > 0\right) \\
&\leq \mathbb{E}\left[\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right)\right] \\
&\underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

as soon as $c > \frac{(d+1)! \cdot d}{2^d}$.

First moment argument

Thus, we have

$$\mathbb{E}\left[\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right)\right] \approx n^d \cdot \exp\left(-\frac{2^d}{(d+1)!} \cdot c \cdot \log(n)\right)$$

as $n \to \infty$, hence

$$\begin{aligned}
\mathbb{P}\left(\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) > k_n(c)\right) &= \mathbb{P}\left(\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right) > 0\right) \\
&\leq \mathbb{E}\left[\#\left(\mathbb{T}_n^d \Big\backslash \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right)\right] \\
&\underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

as soon as $c > \frac{(d+1)! \cdot d}{2^d}$.

The actual threshold is $c_* = \frac{d! \cdot d}{2^d}$, as given by the second moment calculation... So we need to improve the previous first moment argument.

Improved first moment argument

Let $\left(\overline{B}(x, r_n)\right)_{x \in A_n}$ be a "smart" covering of $\mathbb{T}_n^d$ by balls of radius $r_n = \left\lfloor n^{d/(d+1)} \right\rfloor$, so that $\#A_n \cdot r_n^d \asymp n^d$, i.e, $\#A_n \asymp n^{d/(d+1)}$.
Now, observe that

$$\left(\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) > k_n(c) + r_n\right) \subset \bigcup_{x \in A_n} \left(x \notin \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right).$$

Improved first moment argument

Let $\left(\overline{B}(x, r_n)\right)_{x \in A_n}$ be a "smart" covering of $\mathbb{T}_n^d$ by balls of radius $r_n = \left\lfloor n^{d/(d+1)} \right\rfloor$, so that $\#A_n \cdot r_n^d \asymp n^d$, i.e, $\#A_n \asymp n^{d/(d+1)}$.
Now, observe that

$$\left(\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) > k_n(c) + r_n\right) \subset \bigcup_{x \in A_n} \left(x \notin \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right).$$

Thus, as soon as $c > \frac{d! \cdot d}{2^d}$, we have

$$\begin{aligned}
&\mathbb{P}\left(\tau_{\mathrm{cc}}\left(\mathbb{T}_n^d\right) > k_n(c) + r_n\right) \\
&\quad \leq \sum_{x \in A_n} \mathbb{P}\left(x \notin \mathcal{B}_n^{\mathrm{cc}}(k_n(c))\right) \\
&\quad \lesssim n^{d/(d+1)} \cdot \exp\left(-\frac{2^d}{(d+1)!} \cdot c \cdot \log(n)\right) \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Rejection sampling random burning of $\mathbb{T}_n^d$

Let

$$\left(\mathcal{B}_n^{\mathrm{rs}}(k) := \bigcup_{i=1}^{k} \overline{B}(Y_i, k-i)\right)_{k\in\mathbb{N}}$$

be the rejection sampling burning process in $\mathbb{T}_n^d$.

Theorem (with A. Contat (2025+))

As $n \to \infty$, we have

$$\sup_{t\in[0,T[} \left| \frac{\#\left(\mathbb{T}_n^d \setminus \mathcal{B}_n^{\mathrm{rs}}\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} - \frac{1}{y^{(d)}(t)} \right| \xrightarrow{\mathbb{P}} 0.$$

Coupling arguments

Idea: Construct an instance of the rejection sampling burning process using rejection sampling.

Coupling arguments

Idea: Construct an instance of the rejection sampling burning process using rejection sampling.

For each $k \in \mathbb{N}$ such that $Y_1, \ldots, Y_k$ have been constructed, construct Y_{k+1} as follows: sample independent uniforms in $\mathbb{T}_n^d$ until you find one that does not belong to $\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i)$, and call it Y_{k+1}.

In particular, Y_1 is just a uniform on $\mathbb{T}_n^d$.

Come up with an arbitrary convention if you do not find such a uniform, e.g, if $\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) = \mathbb{T}_n^d$.

Coupling arguments

Idea: Construct an instance of the rejection sampling burning process using rejection sampling.

For each $k \in \mathbb{N}$ such that $Y_1, \dots, Y_k$ have been constructed, construct Y_{k+1} as follows: sample independent uniforms in $\mathbb{T}_n^d$ until you find one that does not belong to $\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i)$, and call it Y_{k+1}.

In particular, Y_1 is just a uniform on $\mathbb{T}_n^d$.

Come up with an arbitrary convention if you do not find such a uniform, e.g, if $\bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) = \mathbb{T}_n^d$.

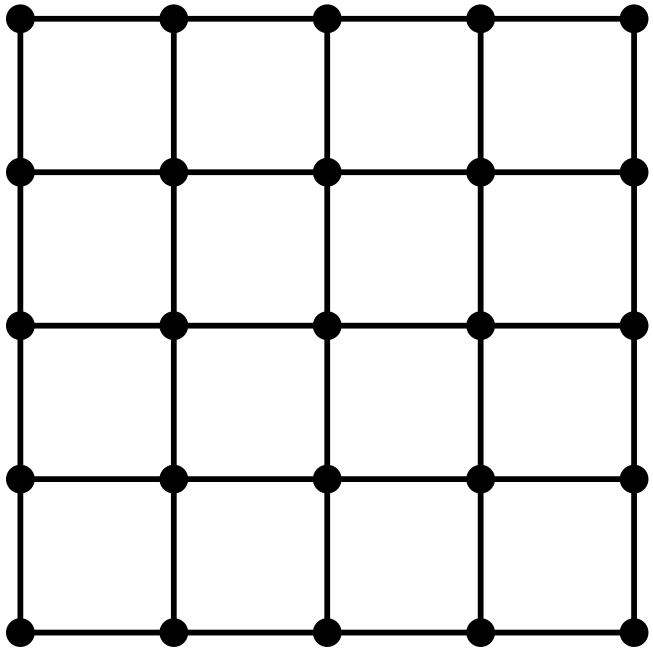
Let

$$\mathcal{B}_n^{\mathrm{rs}}(k) := \bigcup_{i=1}^{k} \overline{B}(Y_i, k-i) \quad \text{for all } k \in \mathbb{N}.$$

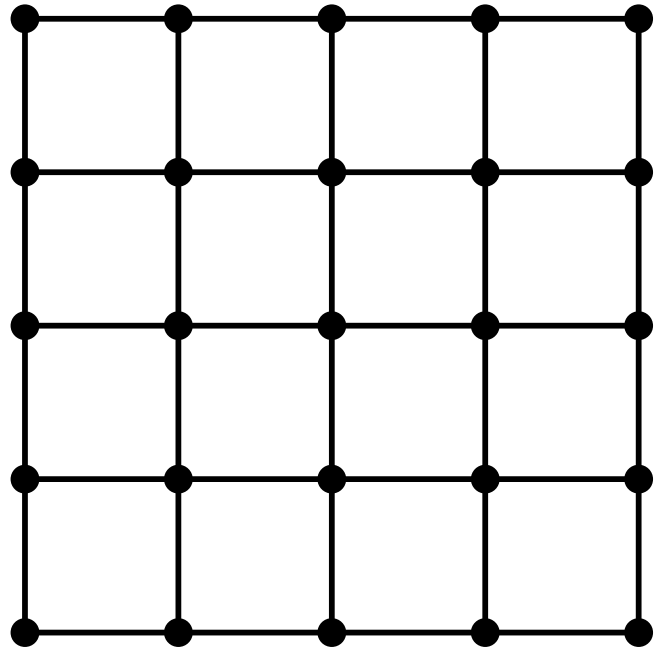
By rejection sampling, the burning process $\left(\mathcal{B}_n^{\mathrm{rs}}(k)\right)_{k\in\mathbb{N}}$ is an instance of the rejection sampling burning process.

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



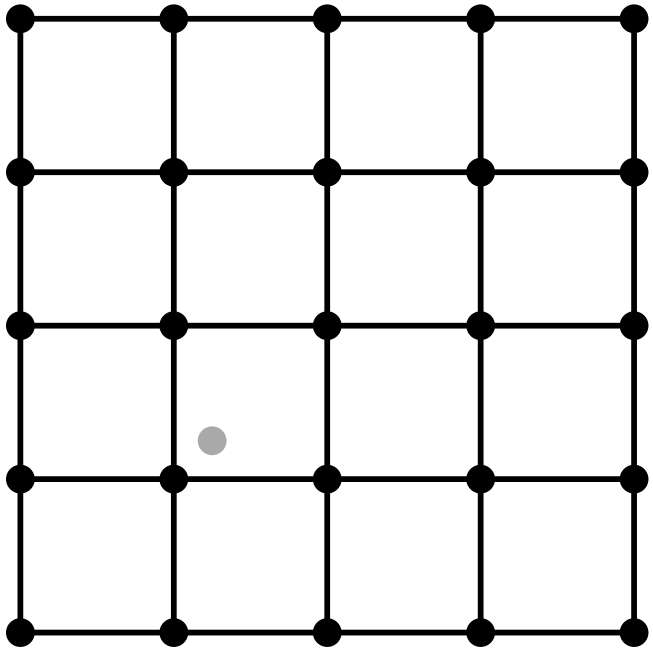
Coupon collector
Step $k = 0$



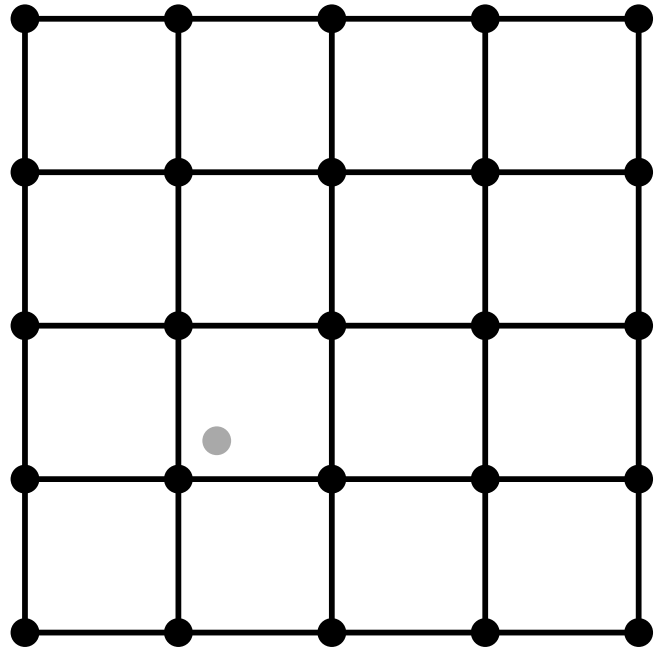
Rejection sampling
Step $k = 0$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



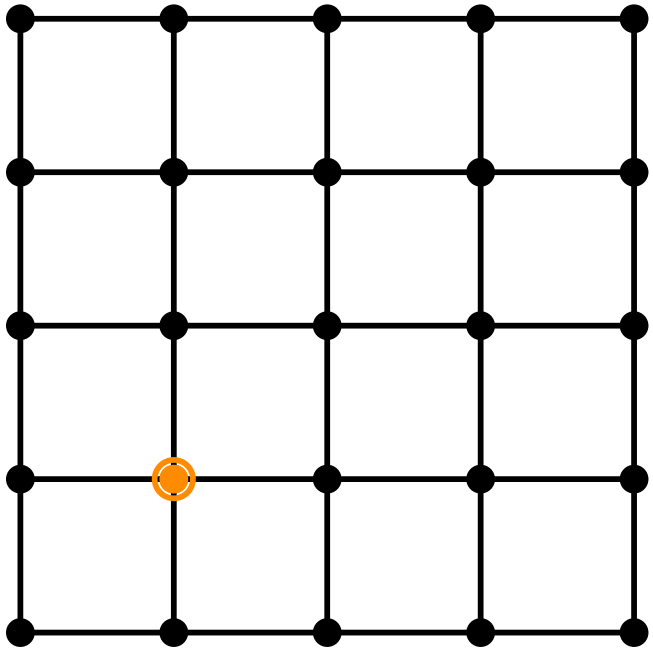
Coupon collector
Step $k = 1$



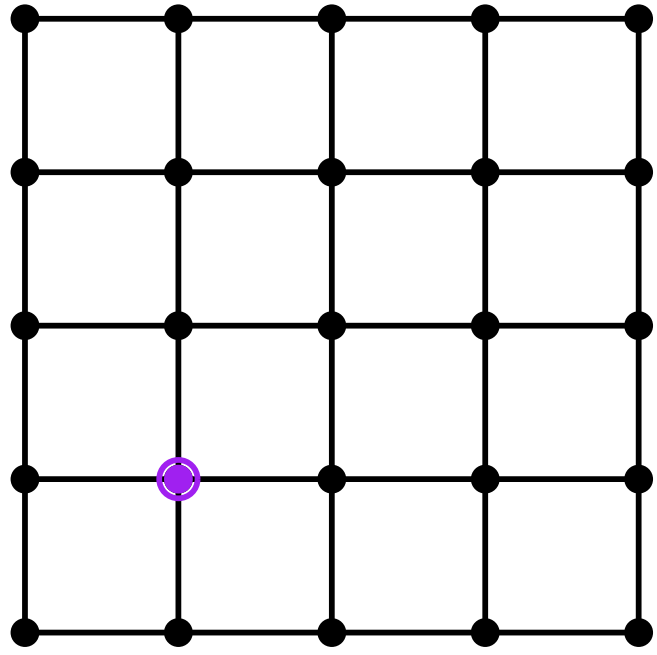
Rejection sampling
Step $k = 1$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



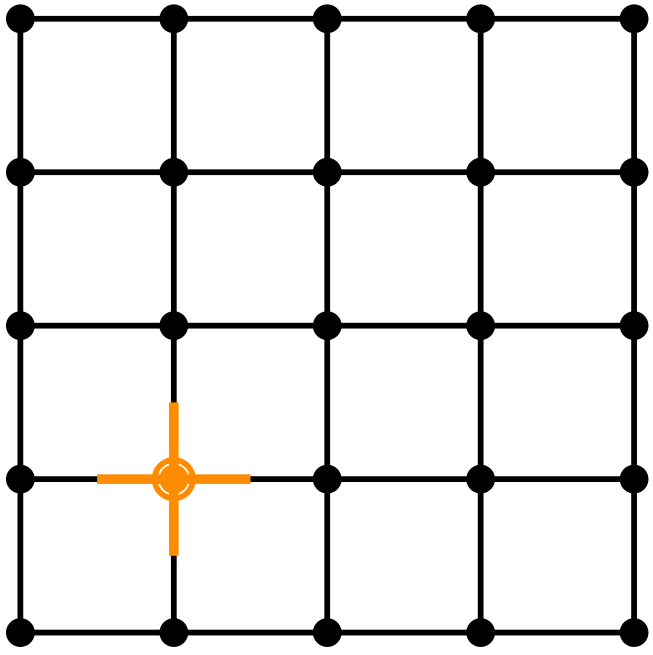
Coupon collector
Step $k = 1$



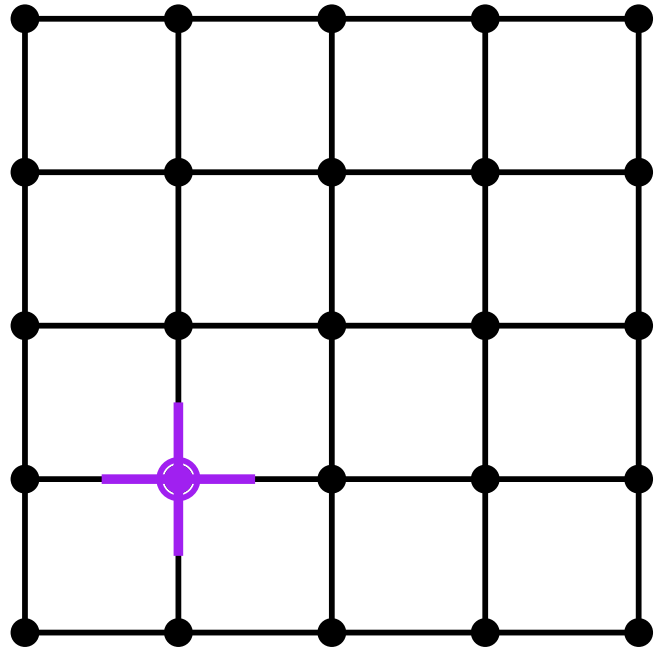
Rejection sampling
Step $k = 1$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



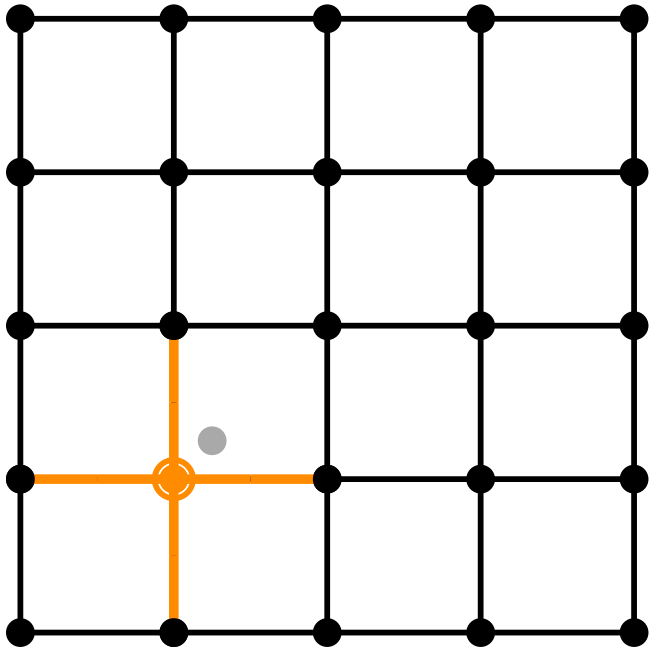
Coupon collector
Step $k = 1 + 1/2$



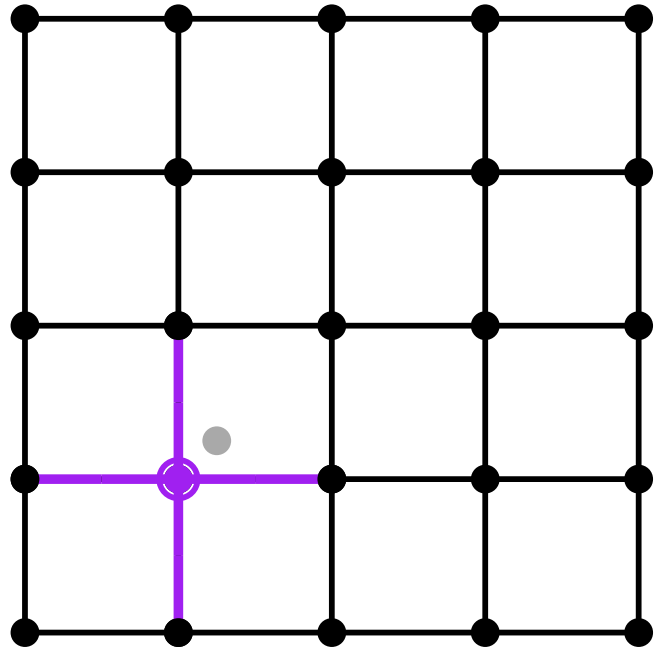
Rejection sampling
Step $k = 1 + 1/2$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



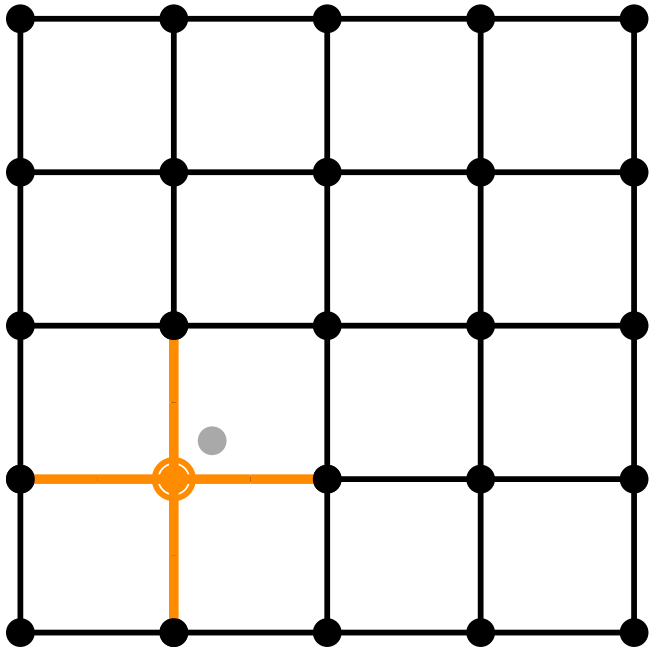
Coupon collector
Step $k = 2^-$



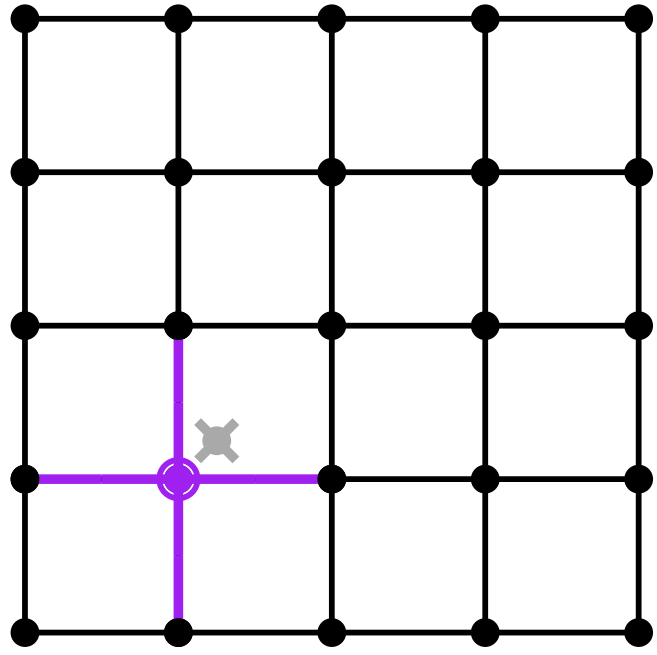
Rejection sampling
Step $k = 2^-$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



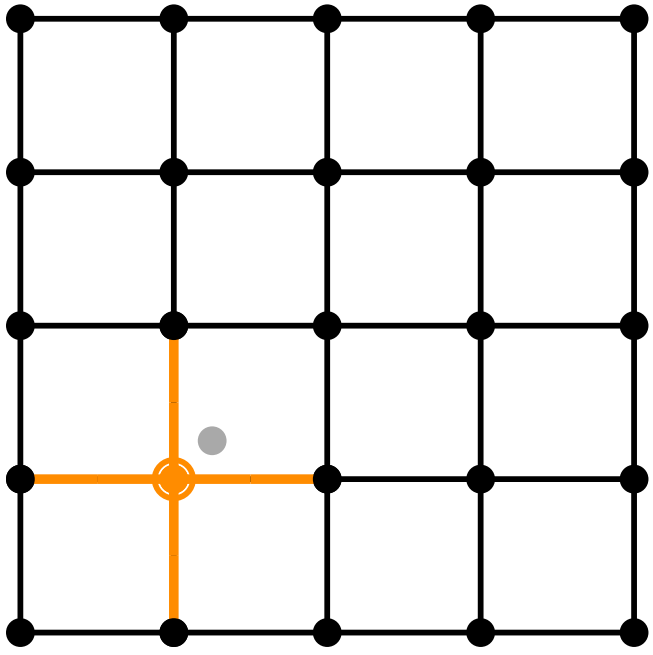
Coupon collector
Step $k = 2^-$



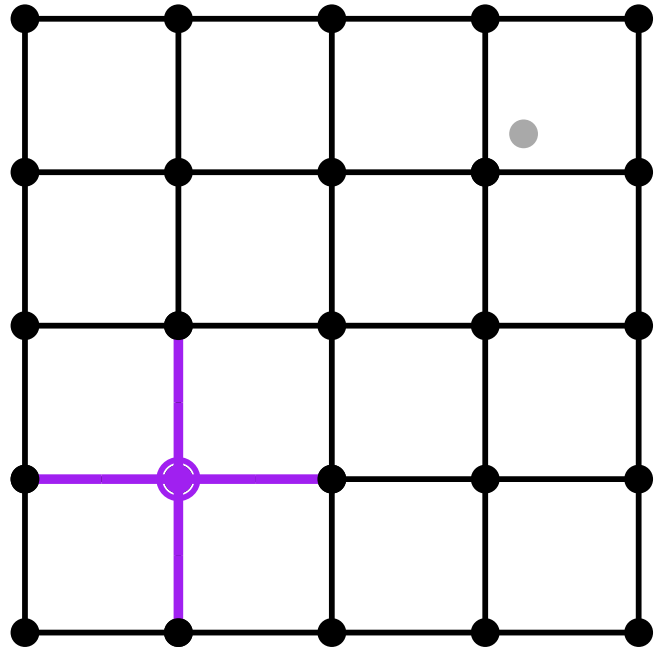
Rejection sampling
Step $k = 2^-$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



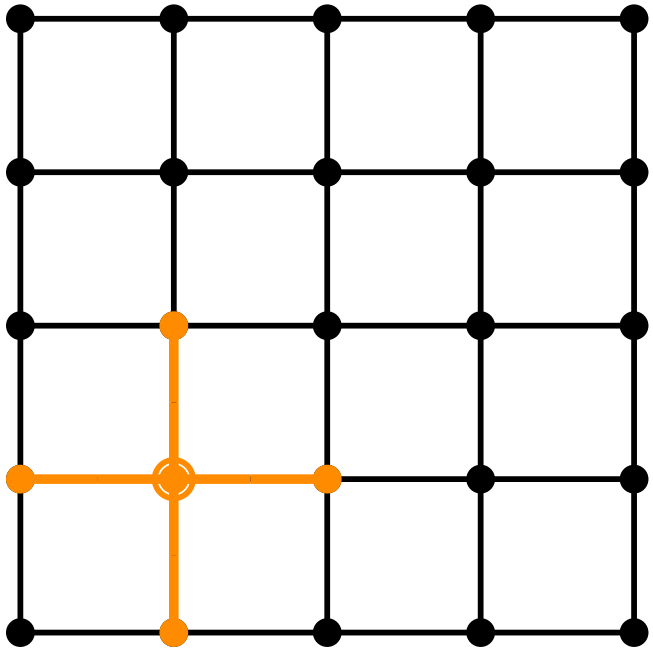
Coupon collector
Step $k = 2^-$



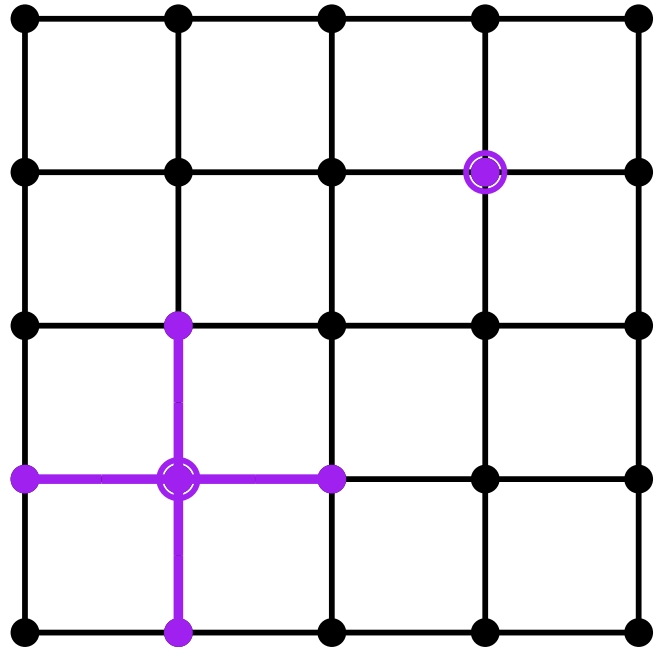
Rejection sampling
Step $k = 2^-$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



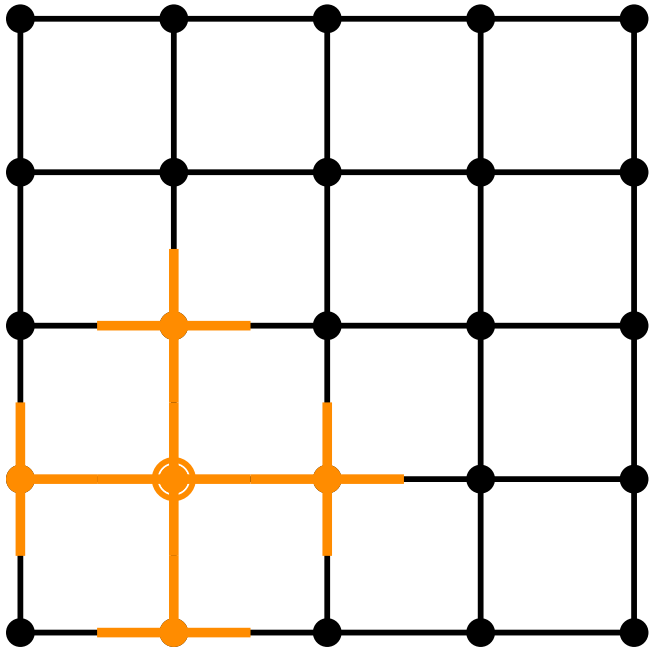
Coupon collector
Step $k = 2$



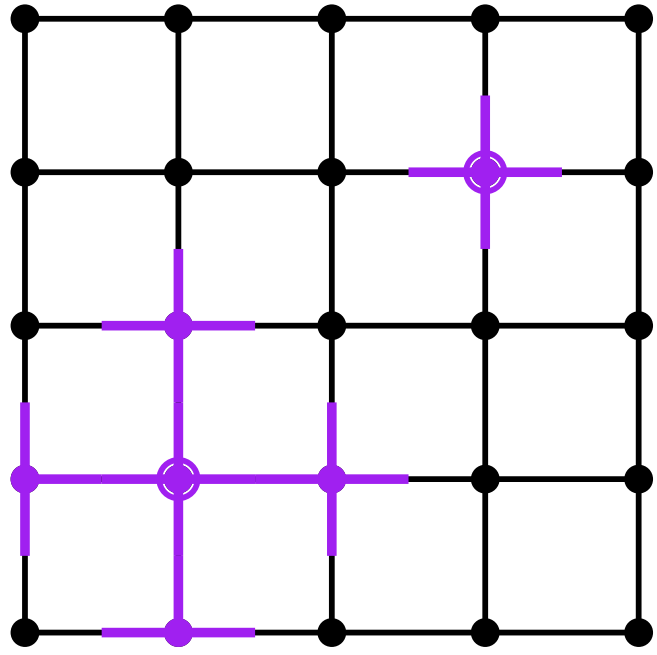
Rejection sampling
Step $k = 2$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



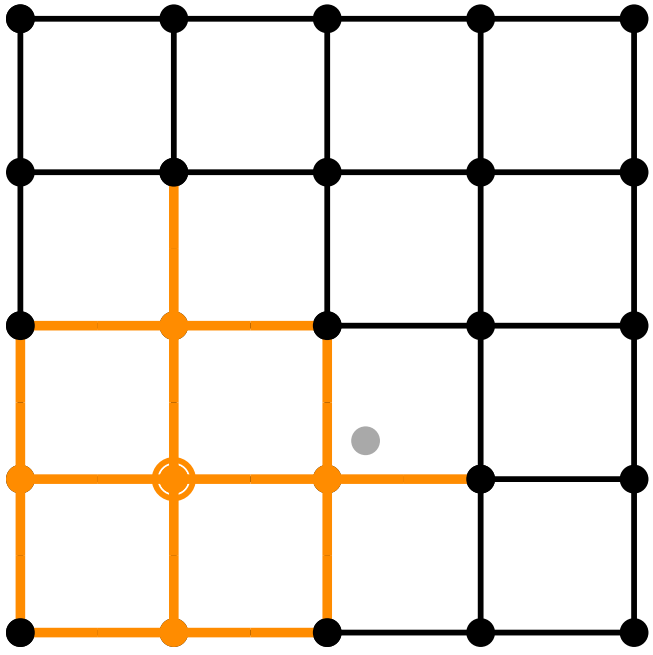
Coupon collector
Step $k = 2 + 1/2$



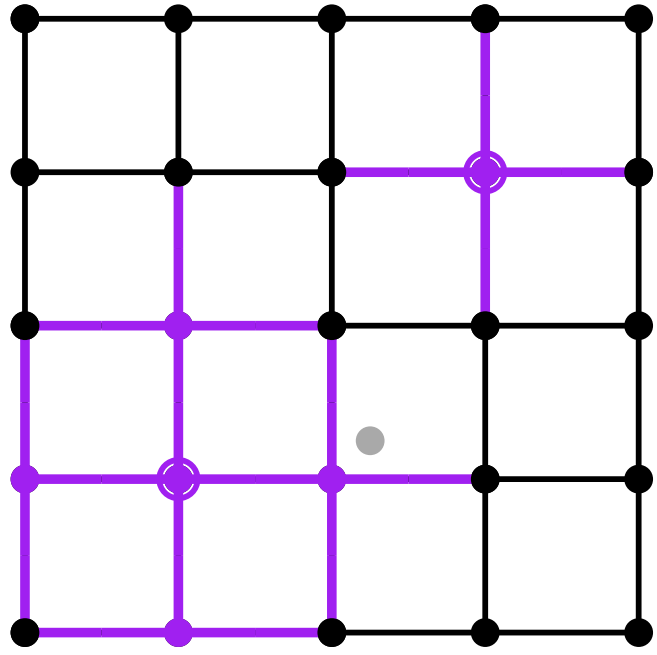
Rejection sampling
Step $k = 2 + 1/2$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



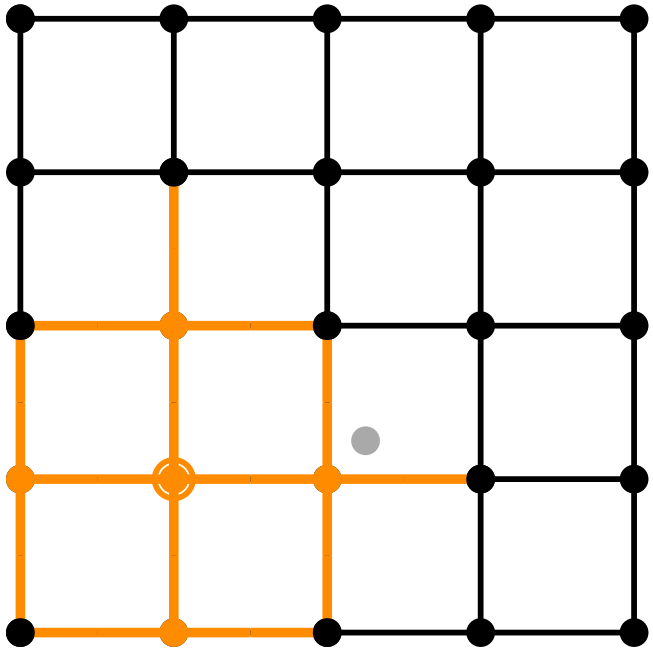
Coupon collector
Step $k = 3^-$



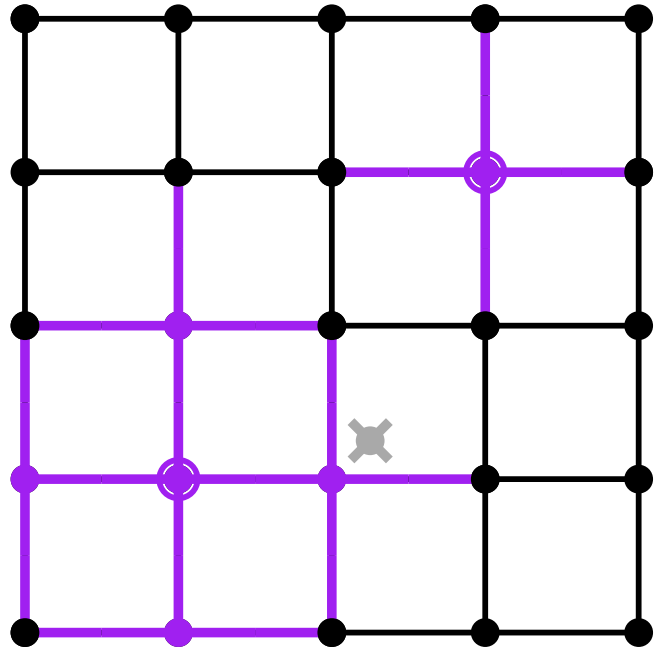
Rejection sampling
Step $k = 3^-$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



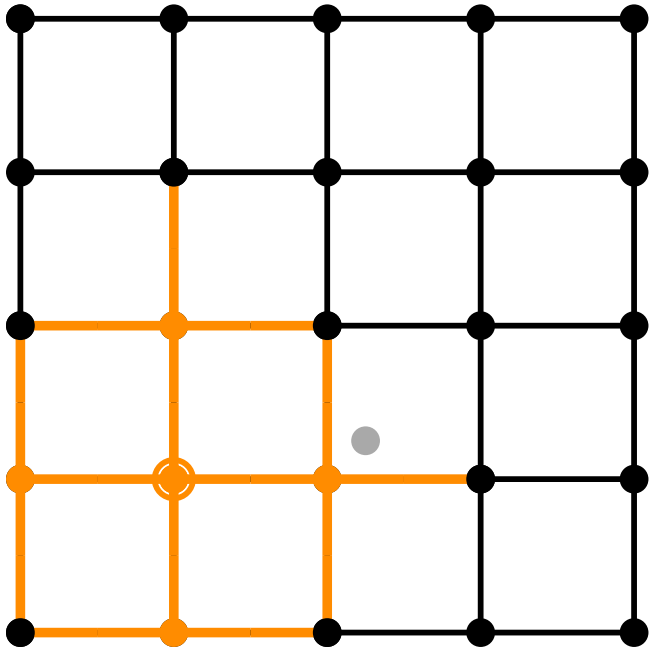
Coupon collector
Step $k = 3^-$



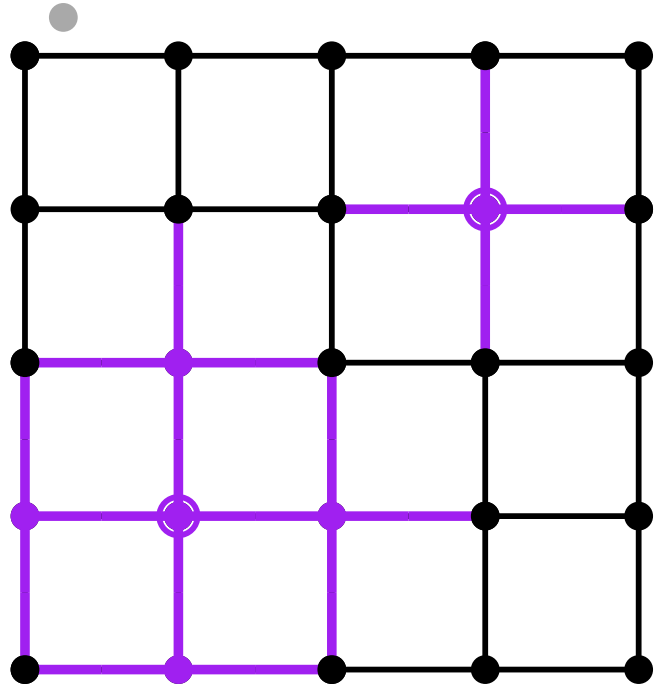
Rejection sampling
Step $k = 3^-$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



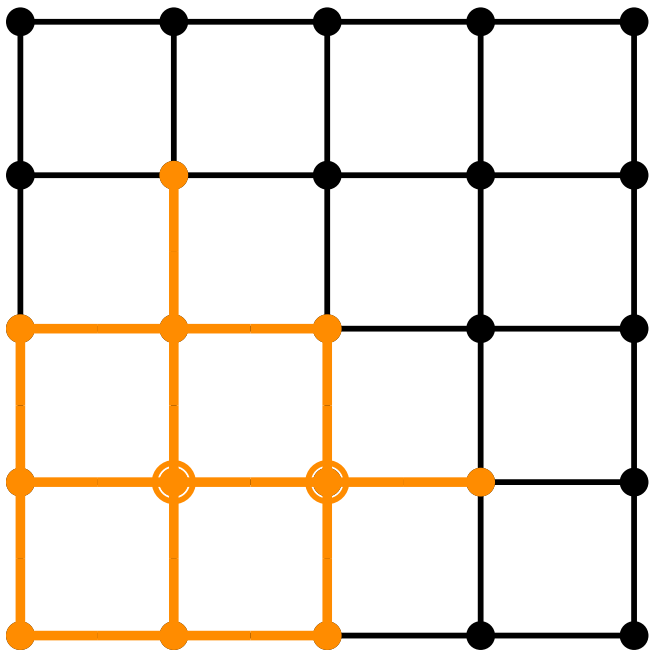
Coupon collector
Step $k = 3^-$



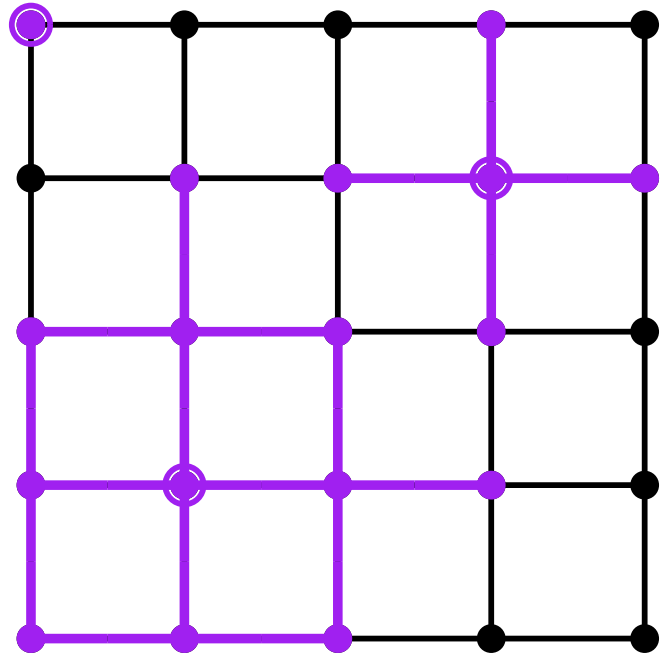
Rejection sampling
Step $k = 3^-$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



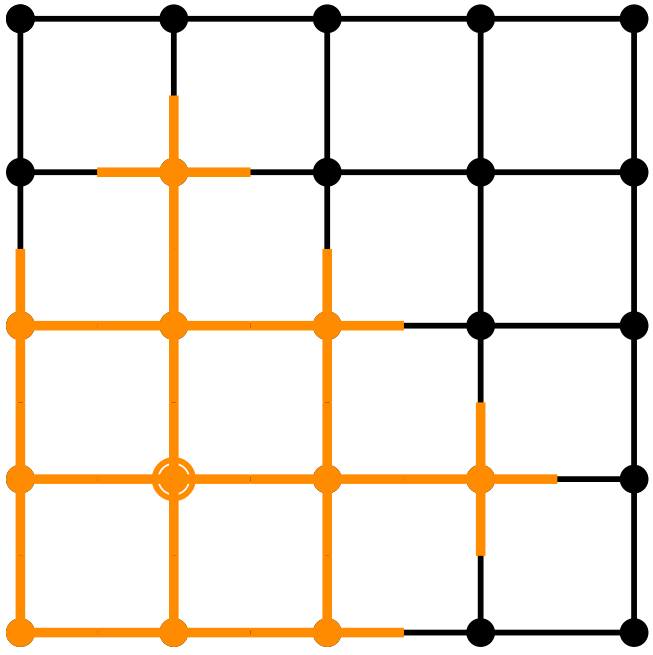
Coupon collector
Step $k = 3$



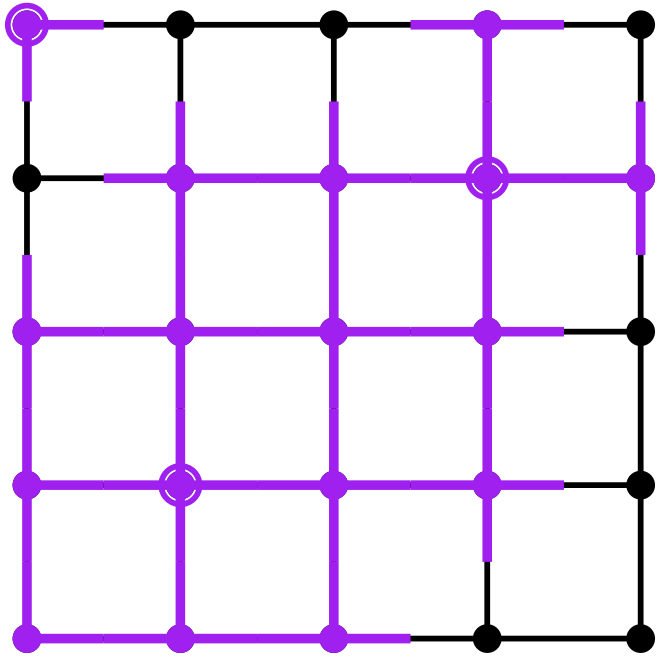
Rejection sampling
Step $k = 3$

Coupling arguments

This construction yields a natural **coupling** of the rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ and the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$.



Coupon collector
Step $k = 3 + 1/2$



Rejection sampling
Step $k = 3 + 1/2$

Coupling arguments

In this coupling, we have $\mathcal{B}_n^{\mathrm{cc}}(k) \subset \mathcal{B}_n^{\mathrm{rs}}(k)$ for all $k \in \mathbb{N}$.

The rejection sampling burning process $\mathcal{B}_n^{\mathrm{rs}}(\cdot)$ is hard to analyse directly...

However, for the coupon collector burning process $\mathcal{B}_n^{\mathrm{cc}}(\cdot)$, it is easy to prove the following scaling limit result:

Proposition

For each $\tau \in \mathbb{R}_+$, we have

$$\sup_{t\in[0,\tau]} \left| \frac{\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} - \exp\left(-\frac{2^d}{(d+1)!} \cdot t^{d+1}\right) \right| \longrightarrow 0 \quad \textit{in probability as } n \to \infty.$$

Coupling arguments

Proof.

By a probabilistic version of Dini's theorem, it suffices to show that for each $t \in \mathbb{R}_+$, we have

$$\frac{\#\left(\mathbb{T}_n^d \setminus \mathcal{B}_n^{\mathrm{cc}}\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} \longrightarrow \exp\left(-\frac{2^d}{(d+1)!} \cdot t^{d+1}\right)$$

in probability as $n \to \infty$.

Coupling arguments

Proof.

By a probabilistic version of Dini's theorem, it suffices to show that for each $t \in \mathbb{R}_+$, we have

$$\frac{\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} \longrightarrow \exp\left(-\frac{2^d}{(d+1)!} \cdot t^{d+1}\right)$$

in probability as $n \to \infty$.

First and second moment argument. Set $k_n := \left\lfloor t \cdot n^{d/(d+1)} \right\rfloor$.

$$\begin{aligned} \frac{\mathbb{E}\left[\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}(k_n)\right)\right]}{n^d} &= \prod_{j=0}^{k_n-1}\left(1 - \frac{\#\overline{B}(0,j)}{n^d}\right) \\ &\approx \exp\left(-\frac{2^d}{(d+1)!} \cdot \frac{k_n^{d+1}}{n^d}\right) \end{aligned}$$

Coupling arguments

Proof.

By a probabilistic version of Dini's theorem, it suffices to show that for each $t \in \mathbb{R}_+$, we have

$$\frac{\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} \longrightarrow \exp\left(-\frac{2^d}{(d+1)!} \cdot t^{d+1}\right)$$

in probability as $n \to \infty$.

First and second moment argument. Set $k_n := \left\lfloor t \cdot n^{d/(d+1)} \right\rfloor$.

$$\frac{\mathbb{E}\left[\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}(k_n)\right)\right]}{n^d} \underset{n\to\infty}{\longrightarrow} \exp\left(-\frac{2^d}{(d+1)!} \cdot t^{d+1}\right),$$

Coupling arguments

Proof.

By a probabilistic version of Dini's theorem, it suffices to show that for each $t \in \mathbb{R}_+$, we have

$$\frac{\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} \longrightarrow \exp\left(-\frac{2^d}{(d+1)!} \cdot t^{d+1}\right)$$

in probability as $n \to \infty$.

First and second moment argument. Set $k_n := \left\lfloor t \cdot n^{d/(d+1)} \right\rfloor$.

$$\frac{\mathbb{E}\left[\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}(k_n)\right)\right]}{n^d} \underset{n\to\infty}{\longrightarrow} \exp\left(-\frac{2^d}{(d+1)!} \cdot t^{d+1}\right),$$

$$\frac{\mathbb{E}\left[\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}(k_n)\right)^2\right]}{\mathbb{E}\left[\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^{\mathrm{cc}}(k_n)\right)\right]^2} \longrightarrow 1 \quad \text{as } n \to \infty.$$

□

Coupling arguments

Let us now call $\mathcal{B}_n^1(\cdot)$ the coupon collector burning process, and let us construct another burning process $\mathcal{B}_n^2(\cdot)$ as follows.

Coupling arguments

Let us now call $\mathcal{B}_n^1(\cdot)$ the coupon collector burning process, and let us construct another burning process $\mathcal{B}_n^2(\cdot)$ as follows.

For each $k \in \mathbb{N}$ such that $Z_1, \ldots, Z_k$ have been constructed, construct Z_{k+1} as follows: **using the same independent uniforms as before,** sample independent uniforms in $\mathbb{T}_n^d$ until you find one that does not belong to $\mathcal{B}_n^1(k)$, and call it Z_{k+1}.

Come up with an arbitrary convention if you do not find such a uniform, e.g, if $\mathcal{B}_n^1(k) = \mathbb{T}_n^d$.

Coupling arguments

Let us now call $\mathcal{B}_n^1(\cdot)$ the coupon collector burning process, and let us construct another burning process $\mathcal{B}_n^2(\cdot)$ as follows.

For each $k \in \mathbb{N}$ such that $Z_1, \ldots, Z_k$ have been constructed, construct Z_{k+1} as follows: **using the same independent uniforms as before,** sample independent uniforms in $\mathbb{T}_n^d$ until you find one that does not belong to $\mathcal{B}_n^1(k)$, and call it Z_{k+1}.

Come up with an arbitrary convention if you do not find such a uniform, e.g, if $\mathcal{B}_n^1(k) = \mathbb{T}_n^d$.

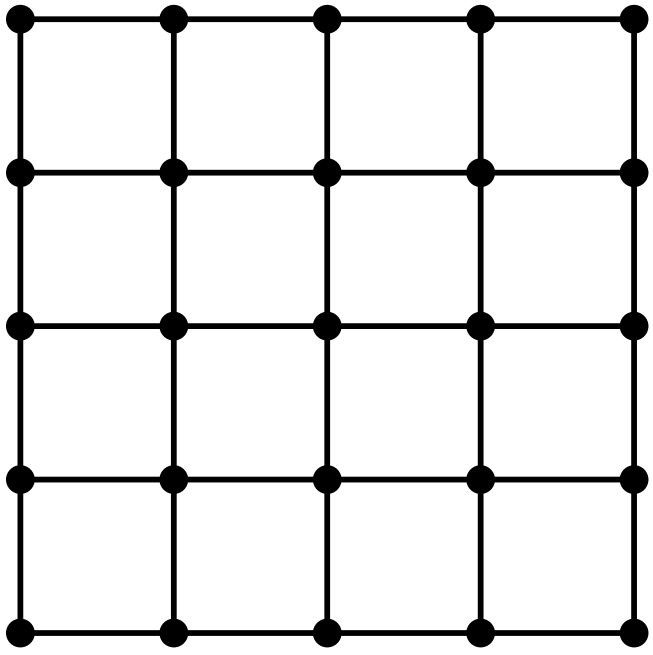
Let

$$\left(\mathcal{B}_n^2(k) := \bigcup_{i=1}^{k} \overline{B}(Z_i, k-i) \right)_{k \in \mathbb{N}}.$$

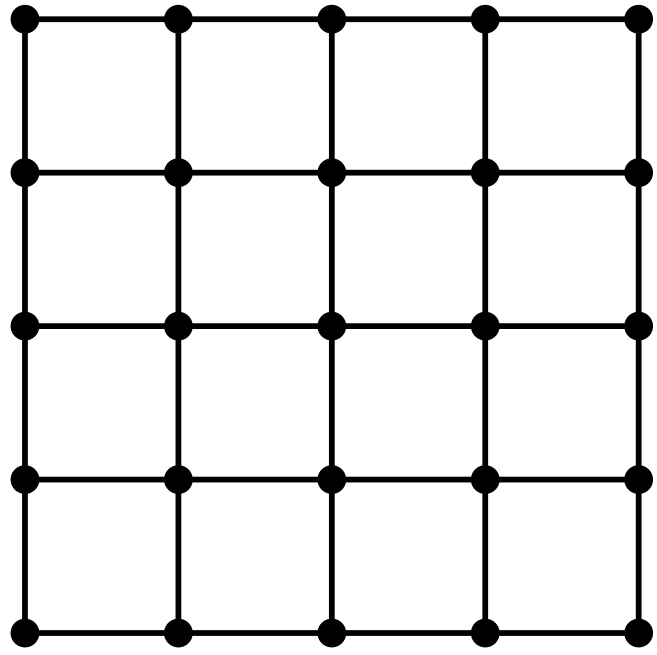
Since $\mathcal{B}_n^1(k) \subset \mathcal{B}_n^{\mathrm{rs}}(k)$ for all $k \in \mathbb{N}$, we get that

$$\mathcal{B}_n^1(k) \subset \mathcal{B}_n^2(k) \subset \mathcal{B}_n^{\mathrm{rs}}(k) \quad \text{for all } k \in \mathbb{N}.$$

Coupling arguments

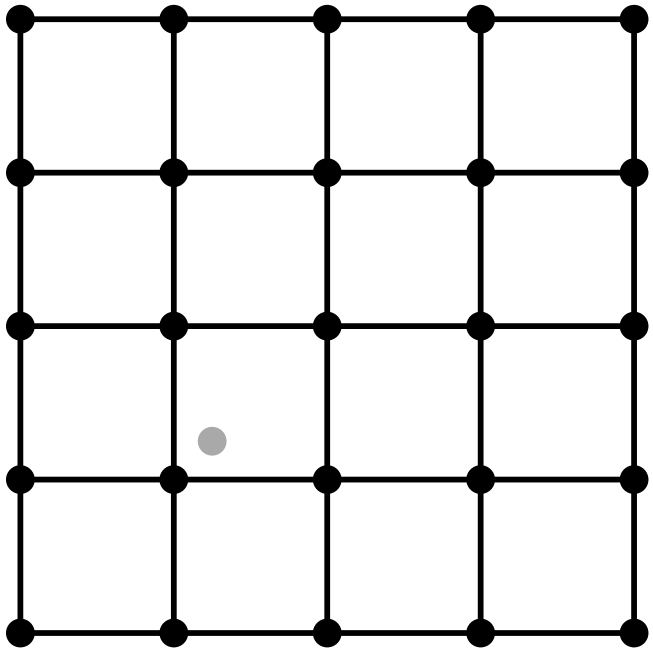


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 0$

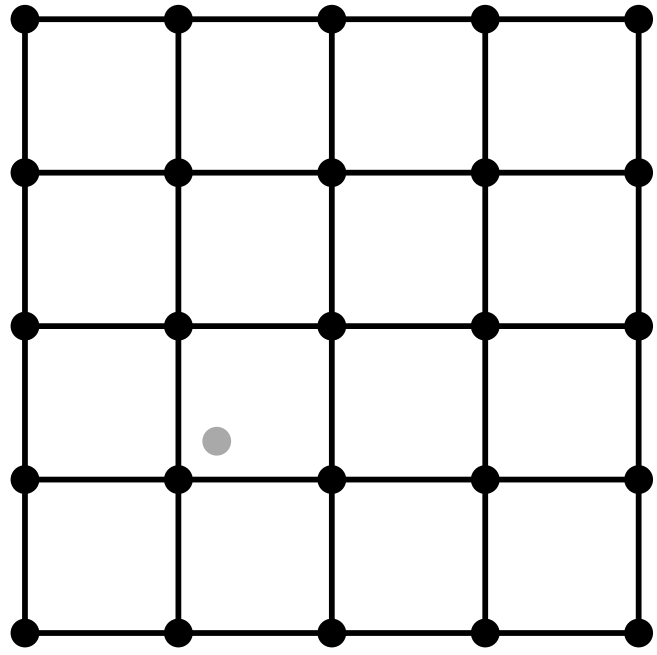


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 0$

Coupling arguments

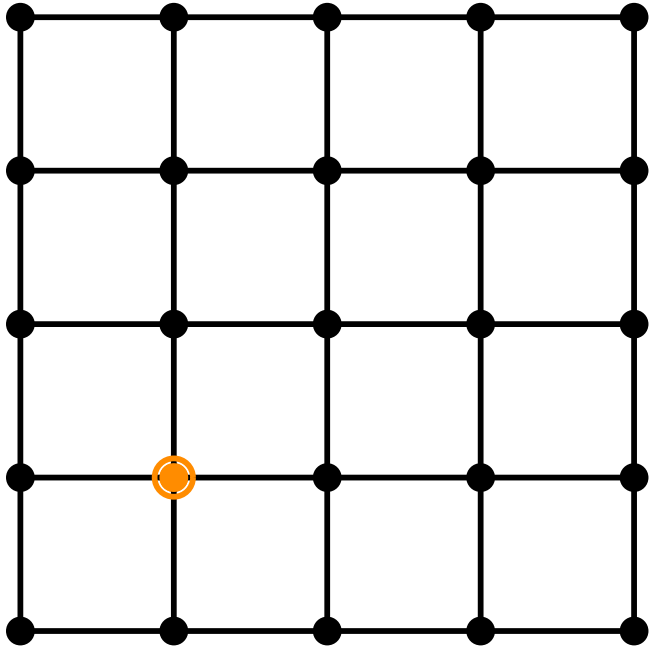


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 1$

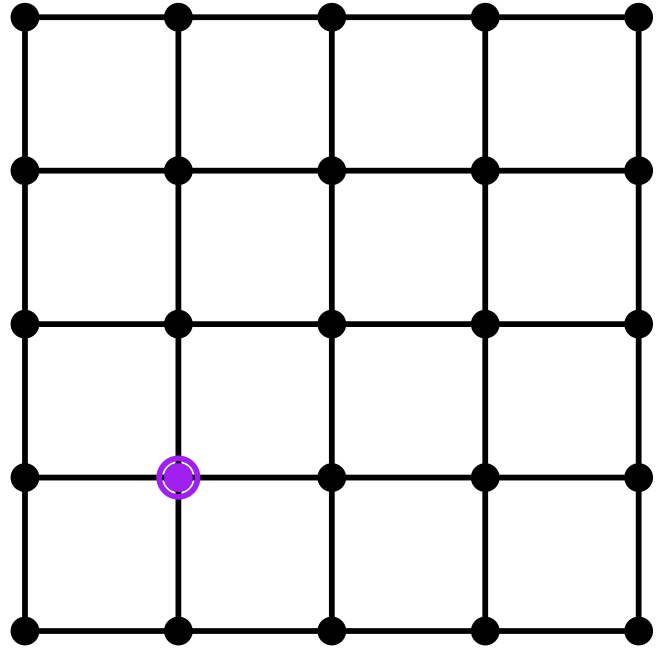


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 1$

Coupling arguments

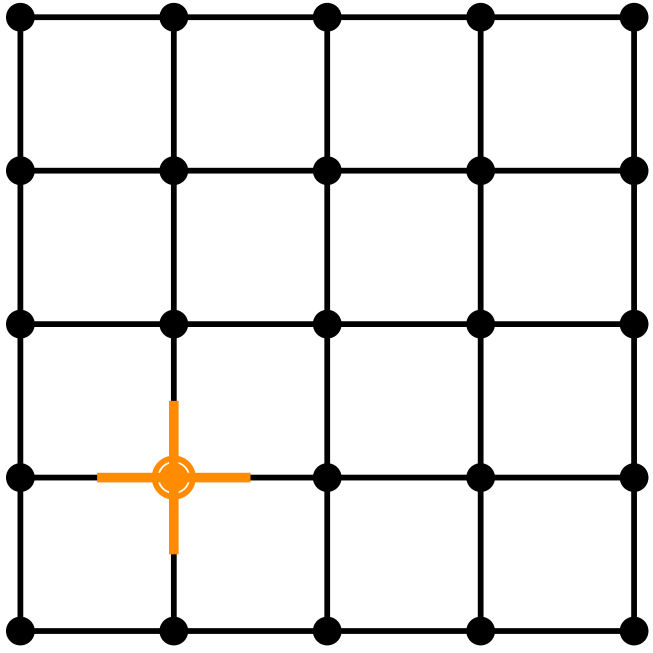


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 1$

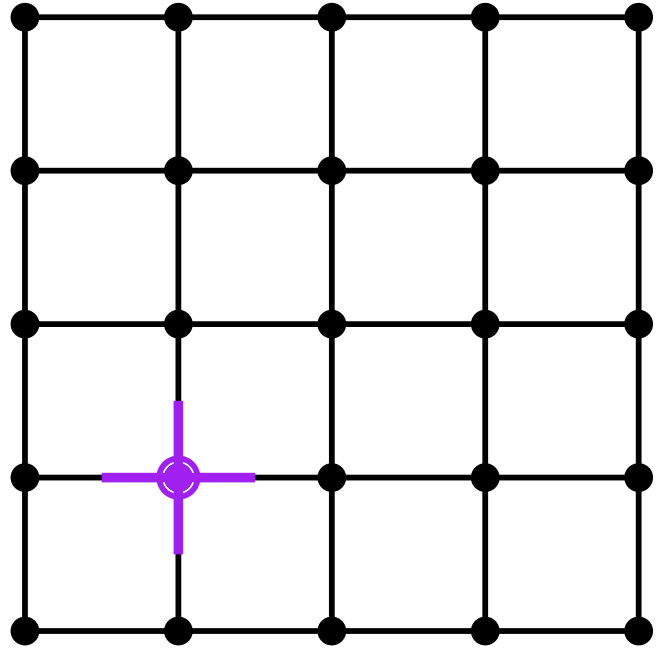


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 1$

Coupling arguments

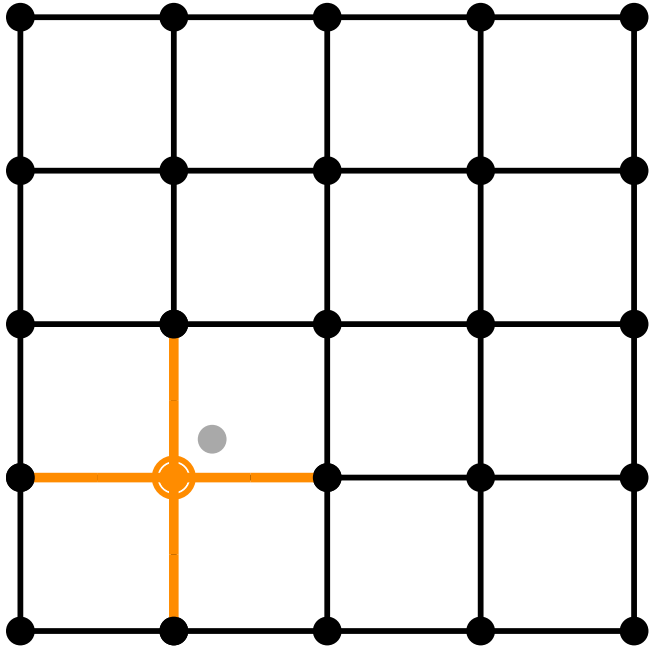


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 1 + 1/2$

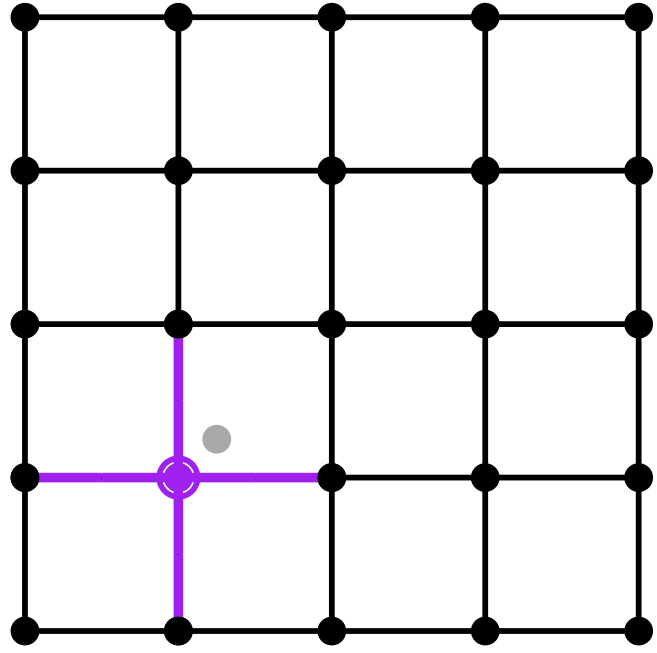


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 1 + 1/2$

Coupling arguments

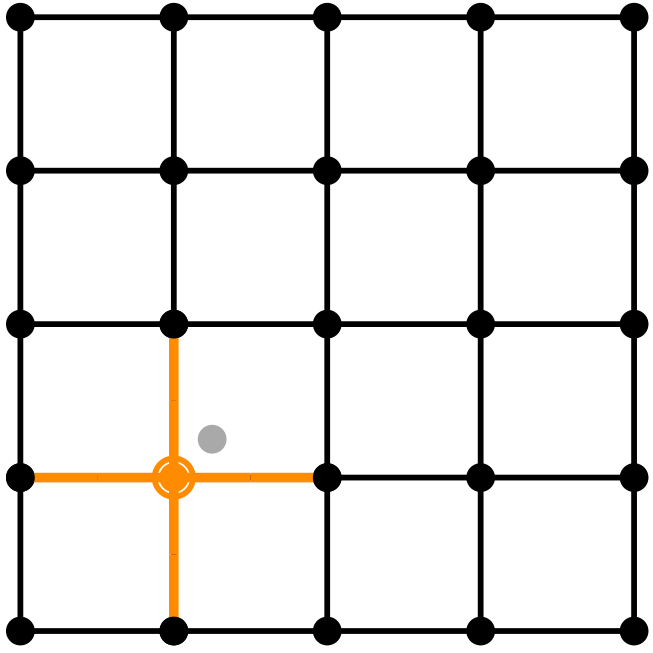


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 2^-$

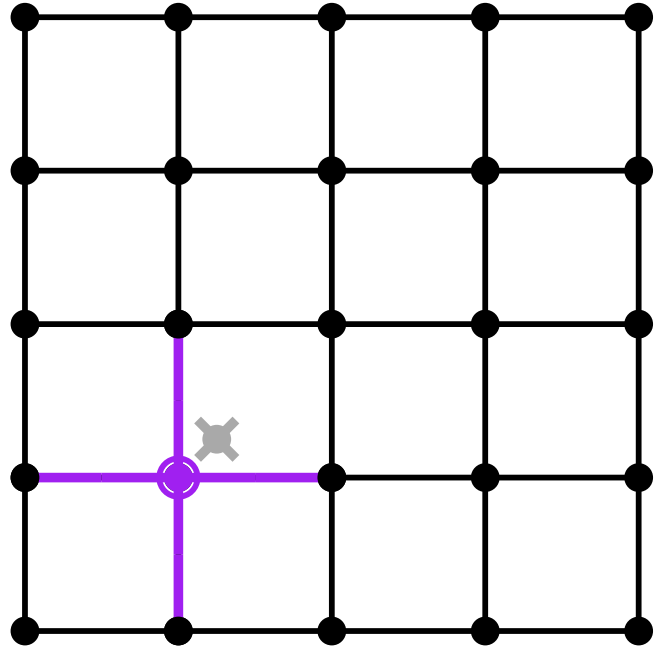


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 2^-$

Coupling arguments

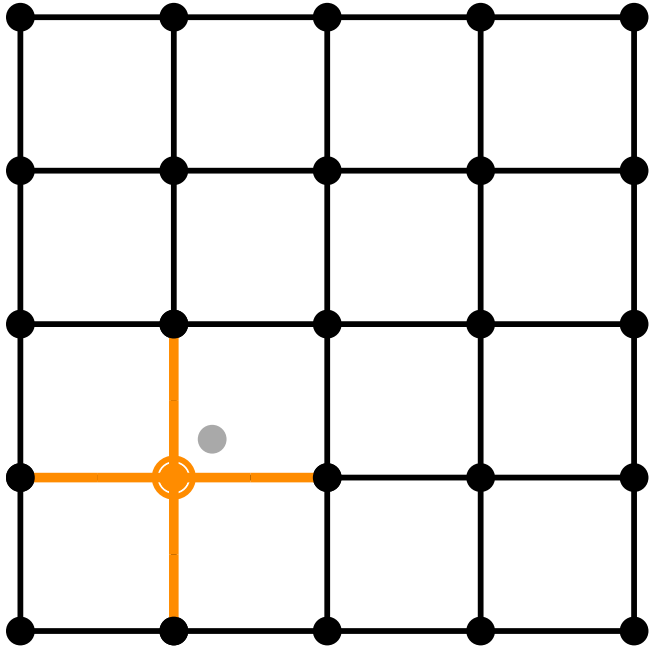


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 2^-$

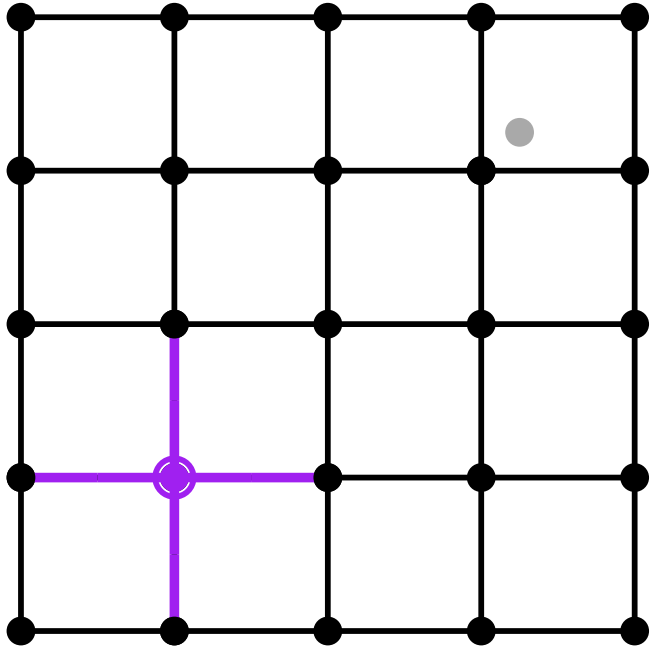


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 2^-$

Coupling arguments

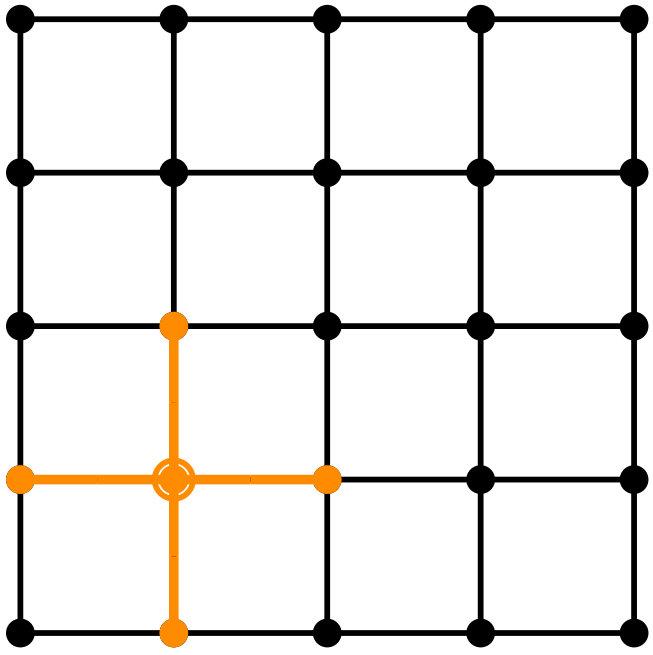


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 2^-$

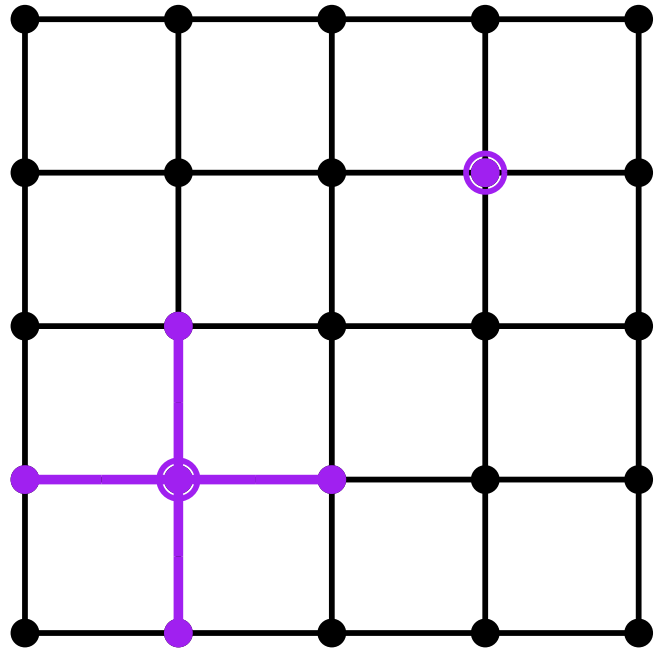


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 2^-$

Coupling arguments

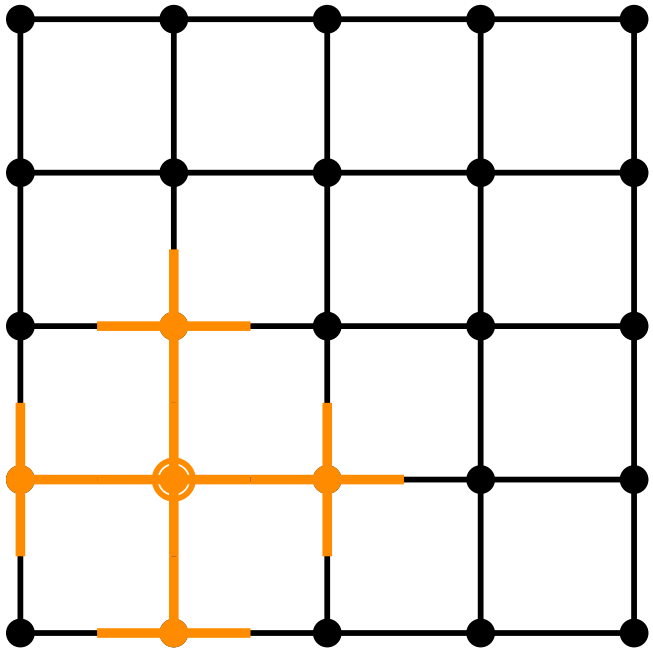


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 2$

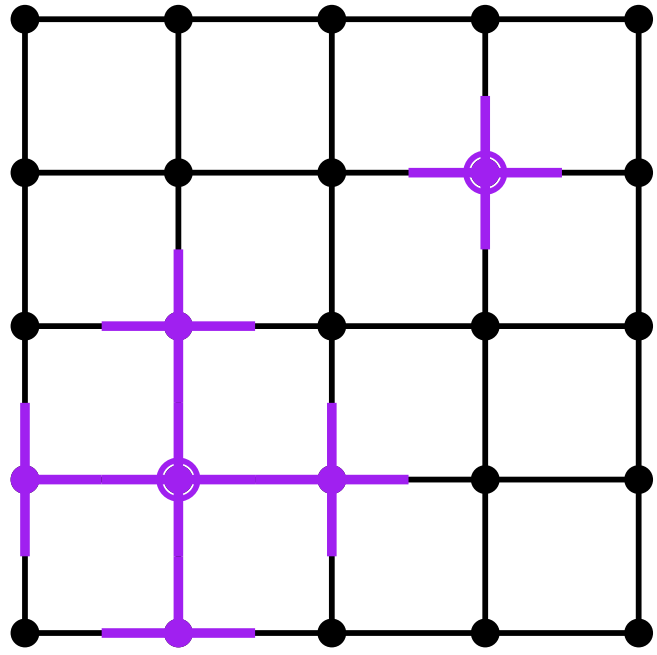


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 2$

Coupling arguments

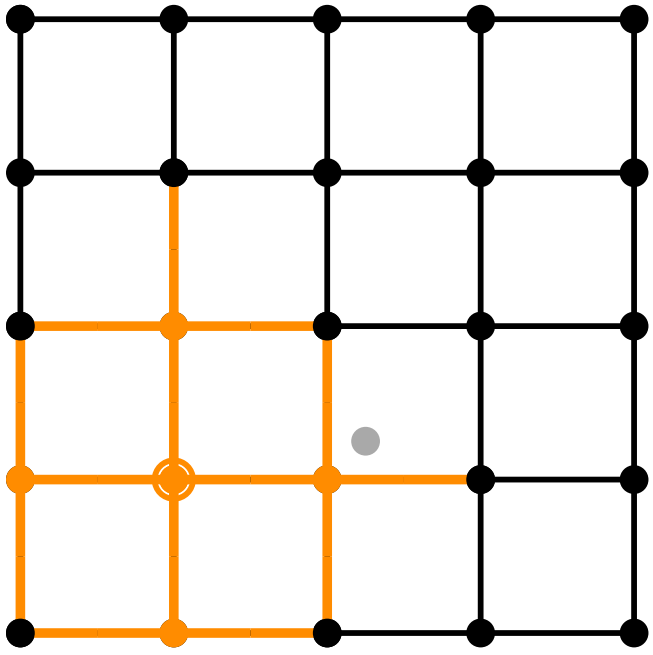


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 2 + 1/2$

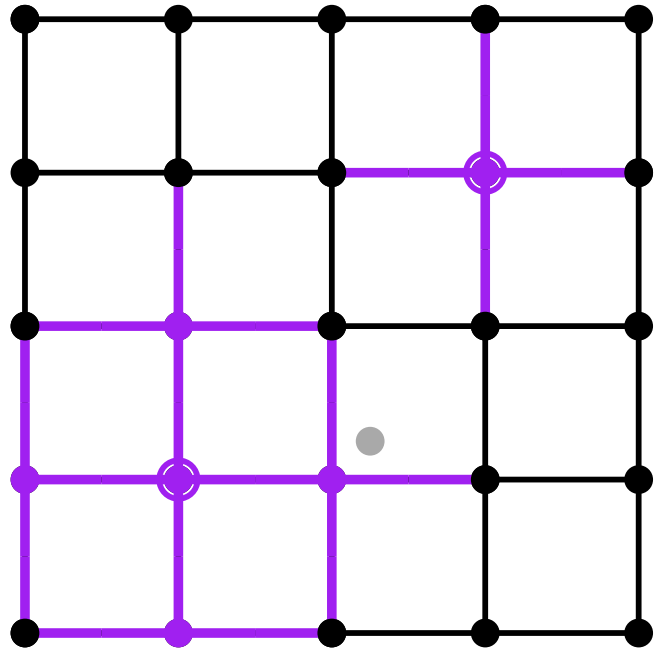


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 2 + 1/2$

Coupling arguments

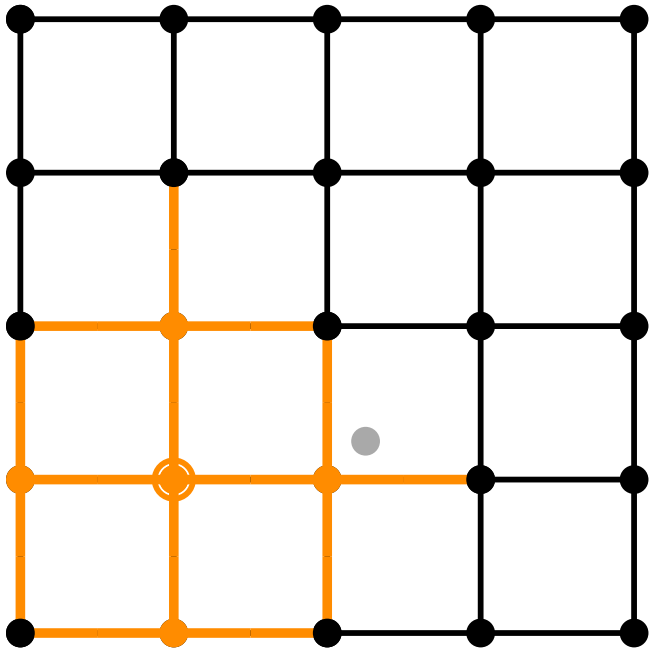


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 3^-$

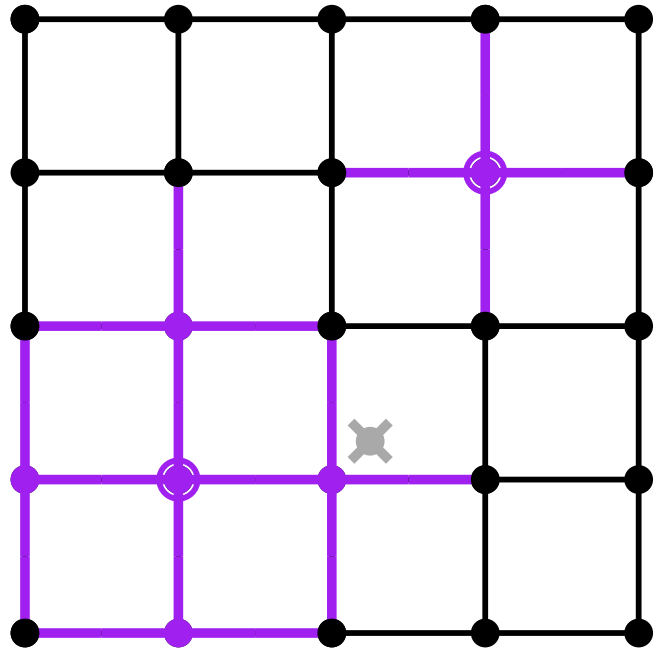


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 3^-$

Coupling arguments

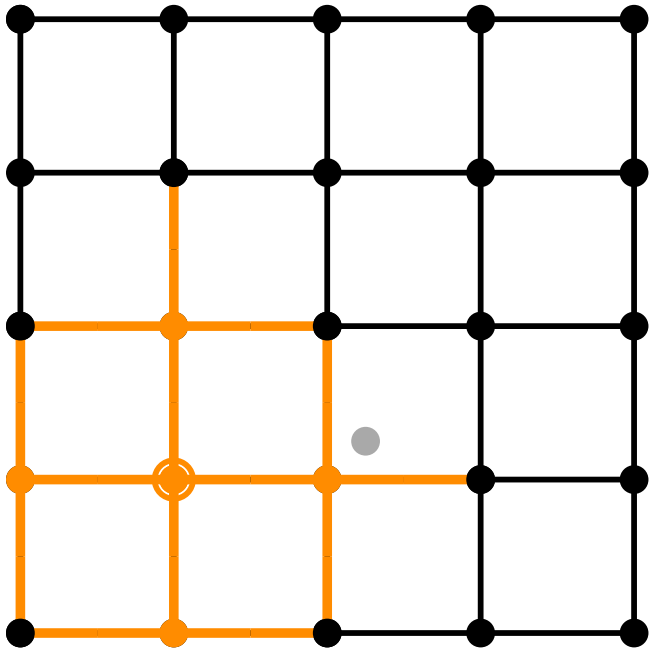


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 3^-$

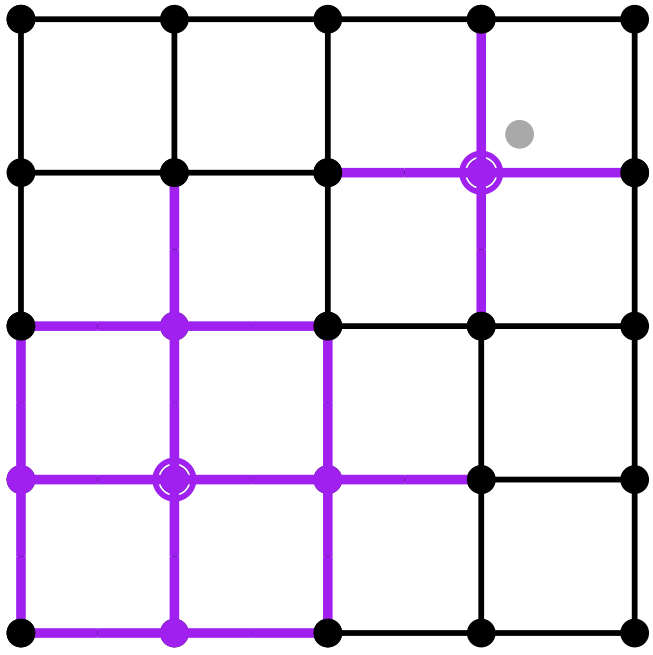


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 3^-$

Coupling arguments

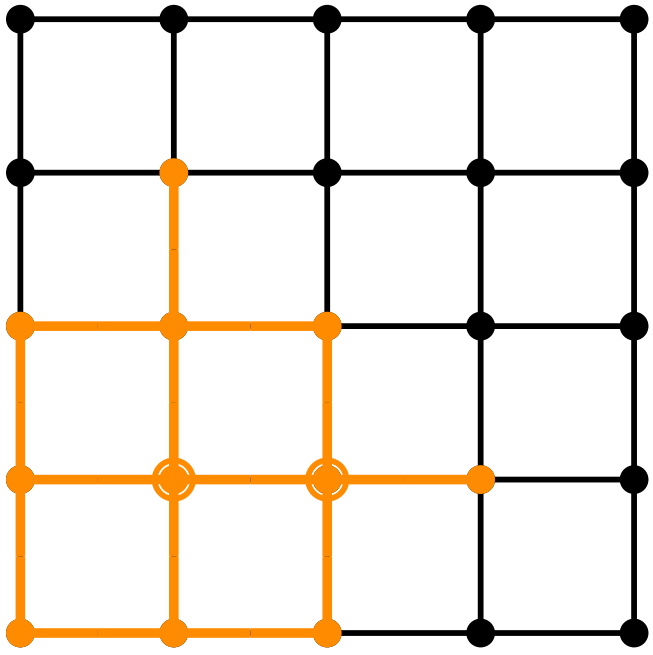


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 3^-$

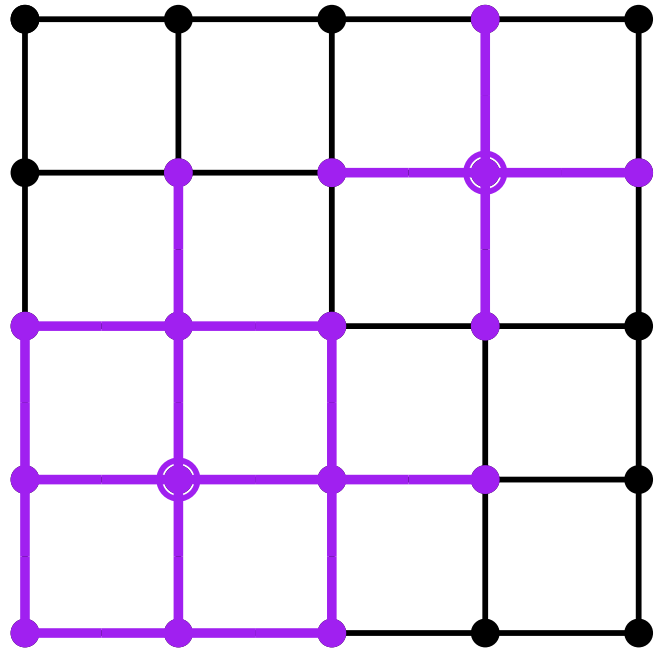


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 3^-$

Coupling arguments

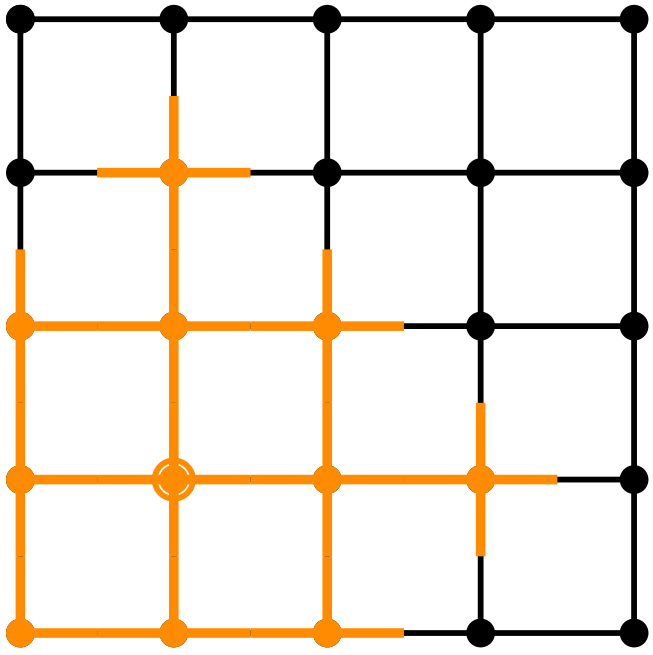


$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 3$

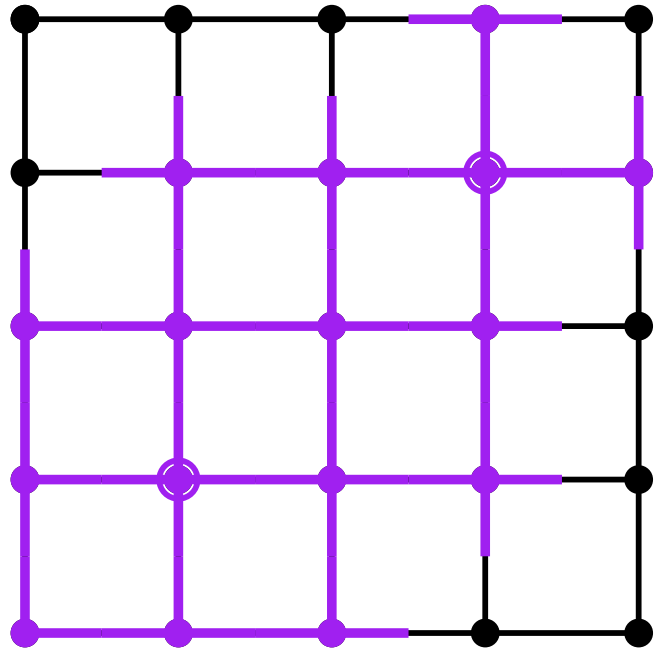


$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 3$

Coupling arguments



$\mathcal{B}_n^1(\cdot)$, coupon collector
Step $k = 3 + 1/2$



$\mathcal{B}_n^2(\cdot)$, not rejection sampling
Step $k = 3 + 1/2$

Coupling arguments

Iterate the procedure to construct a collection $\left(\mathcal{B}_n^p(\cdot)\right)_{p\in\mathbb{N}^*}$ of **intermediate burning processes:** by induction, for each $p \in \mathbb{N}^*$ such that $\mathcal{B}_n^1(\cdot), \ldots, \mathcal{B}_n^p(\cdot)$ have been constructed, construct $\mathcal{B}_n^{p+1}(\cdot)$ as follows.

Coupling arguments

Iterate the procedure to construct a collection $\left(\mathcal{B}_n^p(\cdot)\right)_{p\in\mathbb{N}^*}$ of **intermediate burning processes:** by induction, for each $p \in \mathbb{N}^*$ such that $\mathcal{B}_n^1(\cdot), \ldots, \mathcal{B}_n^p(\cdot)$ have been constructed, construct $\mathcal{B}_n^{p+1}(\cdot)$ as follows.

For each $k \in \mathbb{N}$ such that $W_1, \ldots, W_k$ have been constructed, construct W_{k+1} as follows: **using the same independent uniforms as before,** sample independent uniforms in $\mathbb{T}_n^d$ until you find one that does not belong to $\mathcal{B}_n^p(k)$, and call it W_{k+1}.

Come up with an arbitrary convention if you do not find such a uniform, e.g, if $\mathcal{B}_n^p(k) = \mathbb{T}_n^d$.

Coupling arguments

Iterate the procedure to construct a collection $\left(\mathcal{B}_n^p(\cdot)\right)_{p\in\mathbb{N}^*}$ of **intermediate burning processes:** by induction, for each $p \in \mathbb{N}^*$ such that $\mathcal{B}_n^1(\cdot), \dots, \mathcal{B}_n^p(\cdot)$ have been constructed, construct $\mathcal{B}_n^{p+1}(\cdot)$ as follows.

For each $k \in \mathbb{N}$ such that $W_1, \dots, W_k$ have been constructed, construct W_{k+1} as follows: **using the same independent uniforms as before,** sample independent uniforms in $\mathbb{T}_n^d$ until you find one that does not belong to $\mathcal{B}_n^p(k)$, and call it W_{k+1}.

Come up with an arbitrary convention if you do not find such a uniform, e.g, if $\mathcal{B}_n^p(k) = \mathbb{T}_n^d$.

Let

$$\left(\mathcal{B}_n^{p+1}(k) := \bigcup_{i=1}^{k} \overline{B}(W_i, k-i)\right)_{k\in\mathbb{N}}.$$

Since $\mathcal{B}_n^p(k) \subset \mathcal{B}_n^{\mathrm{rs}}(k)$ for all $k \in \mathbb{N}$, we get that

$$\mathcal{B}_n^p(k) \subset \mathcal{B}_n^{p+1}(k) \subset \mathcal{B}_n^{\mathrm{rs}}(k) \quad \text{for all } k \in \mathbb{N}.$$

Coupling arguments

For all $k \in \mathbb{N}$, we have

$$\mathcal{B}_n^1(k) \subset \mathcal{B}_n^2(k) \subset \ldots \subset \mathcal{B}_n^{\mathrm{rs}}(k).$$

Thus, the intermediate burning processes $\left(\mathcal{B}_n^p(\cdot)\right)_{p \in \mathbb{N}^*}$ interpolate between the coupon collector burning process (for $p = 1$) and the rejection sampling burning process (as $p \to \infty$).

Coupling arguments

For all $k \in \mathbb{N}$, we have

$$\mathcal{B}_n^1(k) \subset \mathcal{B}_n^2(k) \subset \ldots \subset \mathcal{B}_n^{\mathrm{rs}}(k).$$

Thus, the intermediate burning processes $\left(\mathcal{B}_n^p(\cdot)\right)_{p\in\mathbb{N}^*}$ interpolate between the coupon collector burning process (for $p = 1$) and the rejection sampling burning process (as $p \to \infty$).

Recall that for each $\tau \in \mathbb{R}_+$, we have

$$\sup_{t\in[0,\tau]} \left| \frac{\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^1\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} - \frac{1}{f_1(t)} \right|$$

where

$$f_1(t) := \exp\left(\frac{2^d}{(d+1)!} \cdot t^{d+1}\right) \quad \text{for all } t \in \mathbb{R}_+.$$

Coupling arguments

Lemma

For each $p \in \mathbb{N}^*$, the following holds: for each $\tau \in \mathbb{R}_+$, we have

$$\sup_{t\in[0,\tau]} \left| \frac{\#\left(\mathbb{T}_n^d \backslash \mathcal{B}_n^p\left(\left\lfloor t \cdot n^{d/(d+1)} \right\rfloor\right)\right)}{n^d} - \frac{1}{f_p(t)} \right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty,$$

where

$$f_1(t) := \exp\left(\frac{2^d}{(d+1)!} \cdot t^{d+1}\right) \quad \text{for all } t \in \mathbb{R}_+,$$

and for each $p \in \mathbb{N}^*$,

$$f_{p+1}(t) := \exp\left(\frac{2^d}{d!} \cdot \int_0^t (t-s)^d \cdot f_p(s)\mathrm{d}s\right) \quad \text{for all } t \in \mathbb{R}_+.$$

Coupling arguments

Recall that for each $p \in \mathbb{N}^*$,

$$f_{p+1}(t) = \exp\left(\frac{2^d}{d!} \cdot \int_0^t (t-s)^d \cdot f_p(s)\mathrm{d}s\right) \quad \text{for all } t \in \mathbb{R}_+.$$

Proposition

For each $t \in [0, T[$, we have $f_p(t) \uparrow y^{(d)}(t)$ as $p \uparrow \infty$, where $y : [0, T[\to \mathbb{R}$ is the maximal solution of the generalised Blasius equation:

$$y^{(d)}(t) = \exp\left(\frac{2^d}{d!} \cdot \int_0^t (t-s)^d \cdot y^{(d)}(s)\mathrm{d}s\right) \quad \textit{for all } t \in [0, T[.$$

Conclusion

This is how (...) we obtain the desired scaling limit result for the rejection sampling burning process:

$$\sup_{t\in[0,T[}\left|\frac{\#\left(\mathbb{T}_n^d\backslash\mathcal{B}_n^{\mathrm{rs}}\left(\left\lfloor t\cdot n^{d/(d+1)}\right\rfloor\right)\right)}{n^d}-\frac{1}{y^{(d)}(t)}\right|\xrightarrow{\mathbb{P}}0\quad\text{as } n\to\infty.$$

Conclusion

This is how (...) we obtain the desired scaling limit result for the rejection sampling burning process:

$$\sup_{t\in[0,T[}\left|\frac{\#\left(\mathbb{T}_n^d\backslash\mathcal{B}_n^{\mathrm{rs}}\left(\left\lfloor t\cdot n^{d/(d+1)}\right\rfloor\right)\right)}{n^d}-\frac{1}{y^{(d)}(t)}\right|\xrightarrow{\mathbb{P}}0\quad\text{as } n\to\infty.$$

The lower bound on the rejection sampling random burning number readily follows: for each $\varepsilon\in\,]0,T]$, we have

$$\begin{aligned}&\mathbb{P}\left(\frac{\tau_{\mathrm{rs}}\left(\mathbb{T}_n^d\right)}{n^{d/(d+1)}}\leq T-\varepsilon\right)\\ &\quad=\quad\mathbb{P}\left(\#\left(\mathbb{T}_n^d\backslash\mathcal{B}_n^{\mathrm{rs}}\left(\left\lfloor(T-\varepsilon)\cdot n^{d/(d+1)}\right\rfloor\right)\right)=0\right)\\ &\quad\underset{n\to\infty}{\longrightarrow}\quad 0.\end{aligned}$$

Conclusion

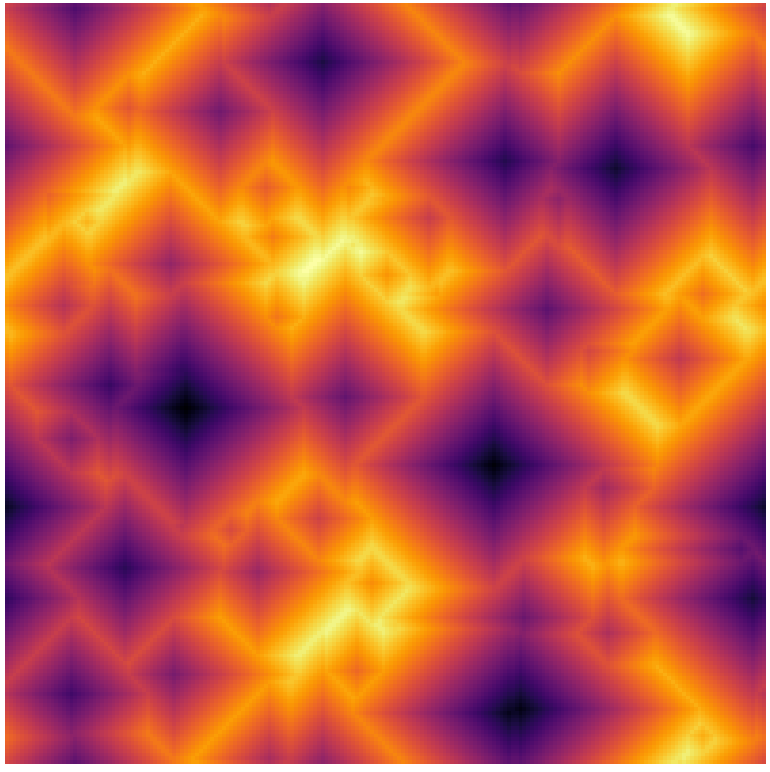
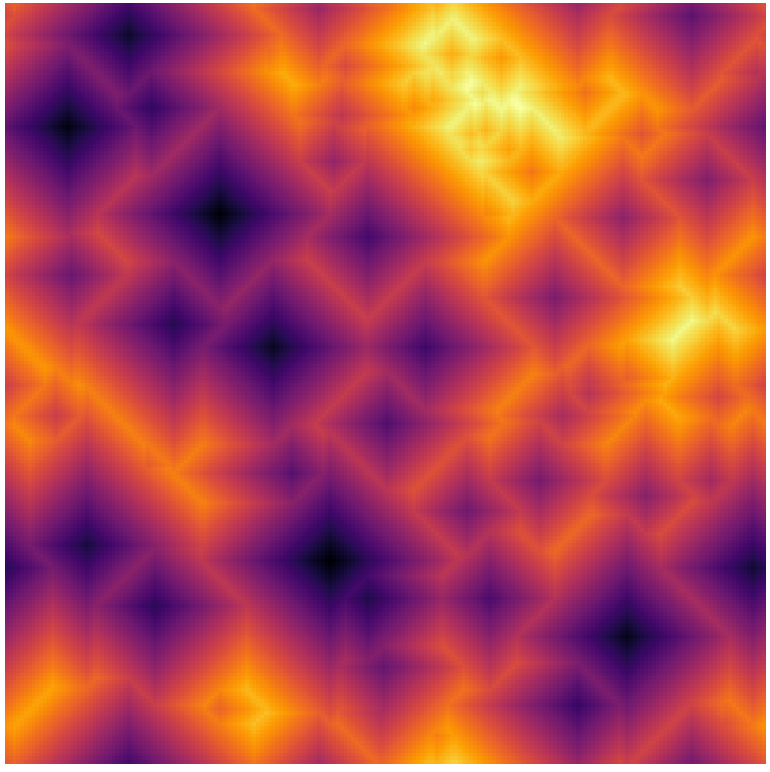
This is how (...) we obtain the desired scaling limit result for the rejection sampling burning process:

$$\sup_{t\in[0,T[}\left|\frac{\#\left(\mathbb{T}_n^d\backslash\mathcal{B}_n^{\mathsf{rs}}\left(\left\lfloor t\cdot n^{d/(d+1)}\right\rfloor\right)\right)}{n^d}-\frac{1}{y^{(d)}(t)}\right|\xrightarrow{\mathbb{P}}0\quad\text{as } n\to\infty.$$

The lower bound on the rejection sampling random burning number readily follows: for each $\varepsilon\in\,]0,T]$, we have

$$\begin{aligned}&\mathbb{P}\left(\frac{\tau_{\mathsf{rs}}\left(\mathbb{T}_n^d\right)}{n^{d/(d+1)}}\leq T-\varepsilon\right)\\ &\quad=\quad\mathbb{P}\left(\#\left(\mathbb{T}_n^d\backslash\mathcal{B}_n^{\mathsf{rs}}\left(\left\lfloor(T-\varepsilon)\cdot n^{d/(d+1)}\right\rfloor\right)\right)=0\right)\\ &\quad\xrightarrow[n\to\infty]{}\quad 0.\end{aligned}$$

The upper bound requires an extra argument...

Thanks for your attention!