Percolation (or not!) for Poissonian systems of stopped paths

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Joint work with **D. Dereudre** (Lille), **J.-B. Gouéré** (Tours)

(started some time ago with **S. Le Stum**)

Plan

1. Introduction
2. Existence result
3. Percolation result

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1 Introduction

2 Existence result

3 Percolation result

Percolation or not?

- $C := \{h : \mathbb{R}_+ \to \mathbb{R}^2 \,;$ continuous with $h(0) = 0\}$.
  μ : probability distribution on C.
- $\bar{\xi}$: Poisson Point Process on $\mathbb{R}^2 \times C$ with intensity measure $dx \otimes \mu$.
  $\bar{\xi} := \{(x, h_x) : x \in \xi\}$ where ξ : projection of $\bar{\xi}$ onto $\mathbb{R}^2$.
- At time $t = 0$ and from each $x \in \xi$,
  the path $x + h_x$ starts to grow.
- Stopping rule:
  When a path hits another one,
  it ceases to grow.
  But the stopping path continues to grow...

Is there an unbounded connected component?

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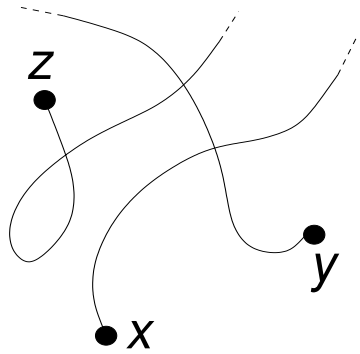

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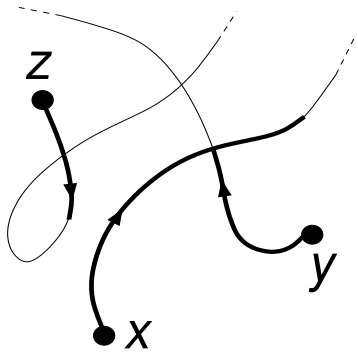

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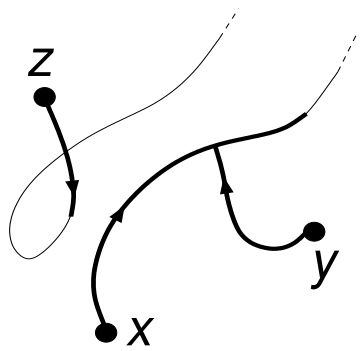

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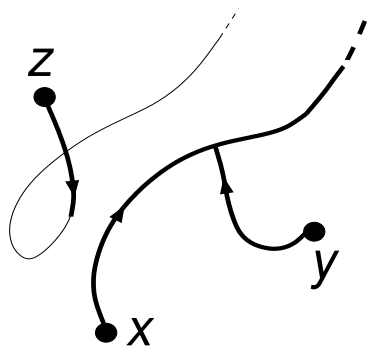

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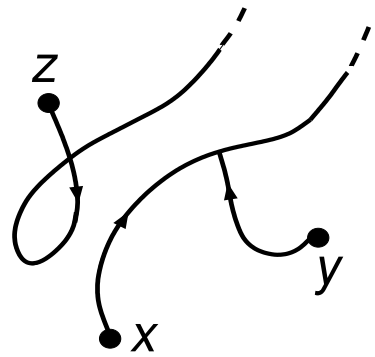

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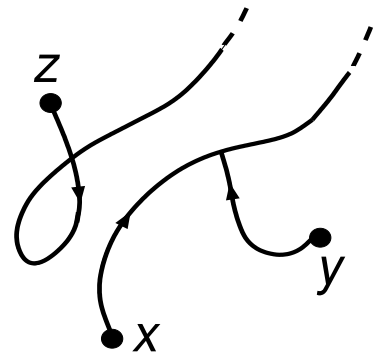

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Notations and remarks

For $(x, h_x) \in \bar{\xi}$:

- $g_x := x + h_x$.
- $G_x(t) := g_x([0,t])$: the path / grain of x until time t.
- $\tau(x) = \tau(x; \bar{\xi}) \in (0, \infty]$: lifetime of the grain of x, i.e. time at which the grain of x is stopped.
  With $\tau(x) = \infty$ if the grain is never stopped.

Main difficulty of the model: strongly non-local.

- Not easy to determine the lifetimes $\tau(x), x \in \xi$.
  I.e. who stops who?
- Very sensitive to local modifications by domino effects:

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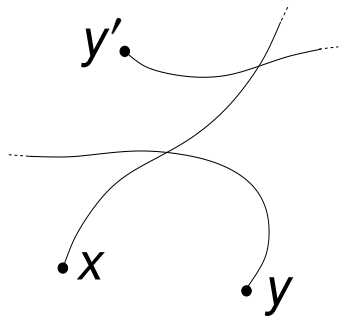

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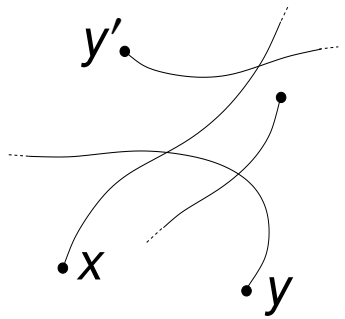

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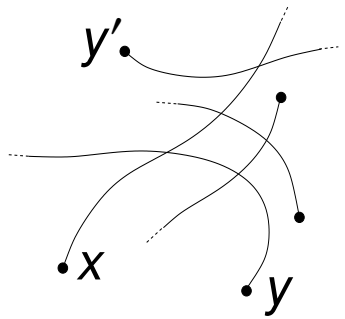

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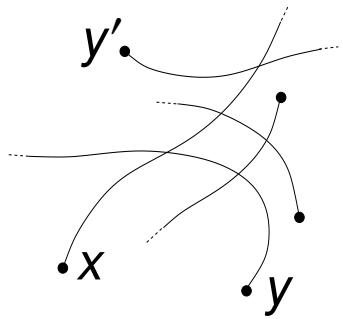

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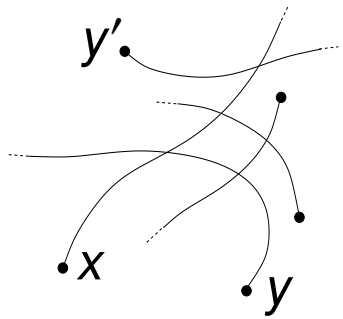

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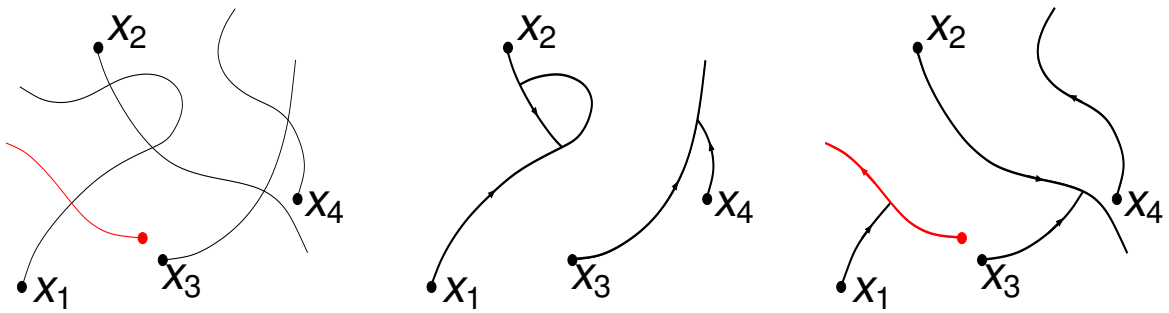

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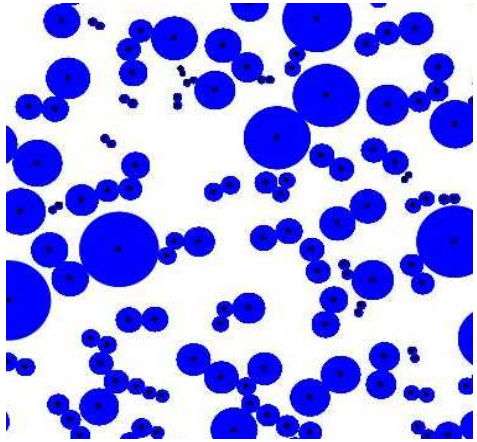

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  At time $t = 0$, balls centered at each $x \in \xi$ grow at speed 1. With the same stopping rule.
  [Häggström, Meester '96]

  ★ The Lilypond model does not percolate a.s.
  (No infinite descending chains in the PPP ξ)

Thin grains:

$\bar{\xi}$: PPP on $\mathbb{R}^2 \times C$ with intensity measure $dx \otimes \mu$ and $h \sim \mu$.

- Line segment model: $h(t) := tV \times U$
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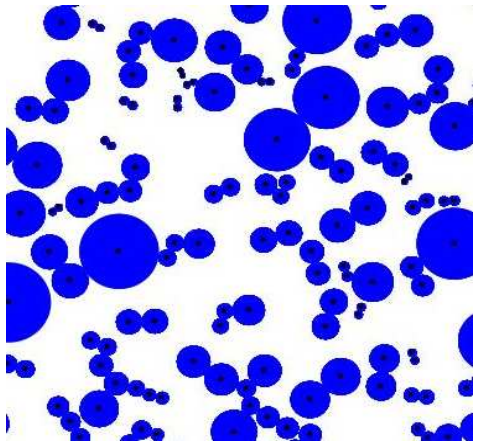

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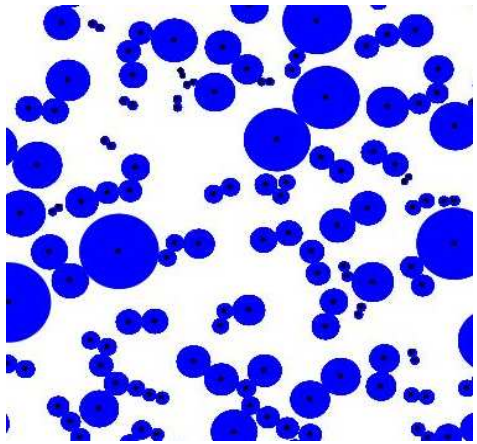

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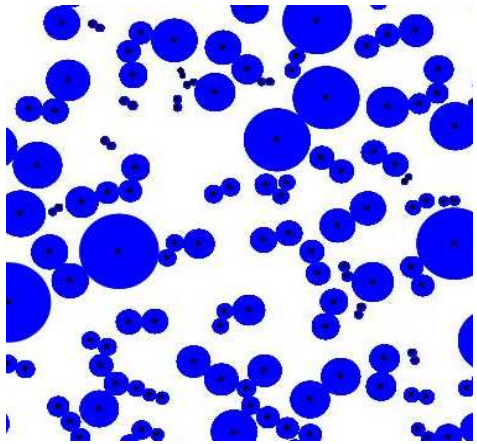

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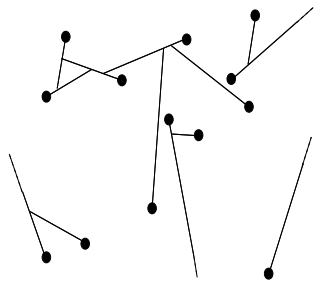

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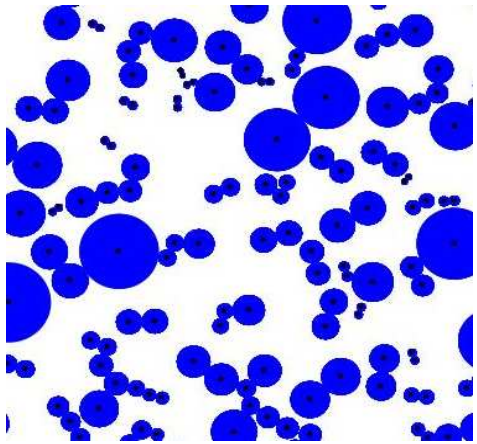

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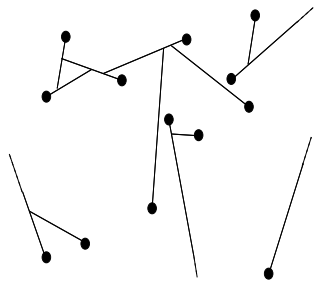

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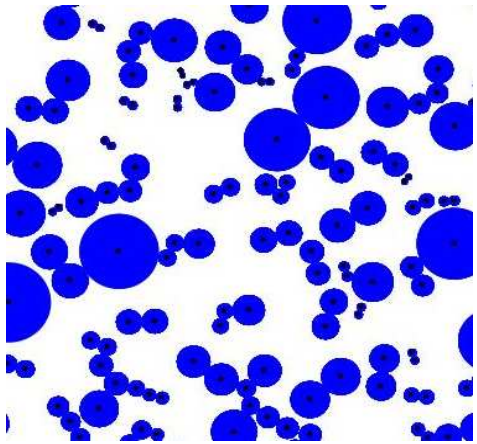

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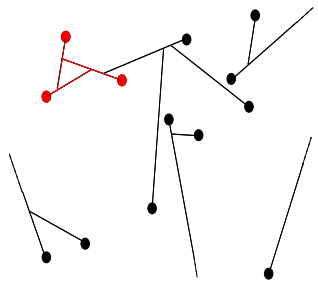

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Existence - Lifetime functions

- $\bar{\mathcal{S}} \subset \mathbb{R}^2 \times \mathcal{C}$: space of configurations (locally finite, simple).
- For $\bar{S} \in \bar{\mathcal{S}}$, $\bar{S} := \{(x, h_x) : x \in S\}$
  where S is the projection of $\bar{S}$ onto $\mathbb{R}^2$.

Dynamical def. for a finite configuration $\bar{S}$.

Static def.: the model is well defined on $\bar{S}$ if $\exists!$ lifetime function
$\tau = \tau(\cdot; \bar{S}) : S \to (0, \infty]$ s.t.

1. *A collision implies a stop.*
   If $\exists x, y \in S, \exists t_x, t_y \geq 0$ s.t. $t_x \leq \tau(x), t_y \leq \tau(y)$
   $g_x(t_x) = g_y(t_y)$ and $t_y \leq t_x$ then $\tau(x) = t_x$.

2. *A stop requires a collision.*
   For any $x \in S$ with $\tau(x) < \infty$ then $\exists y \in S, \exists t_x, t_y \geq 0$ s.t.
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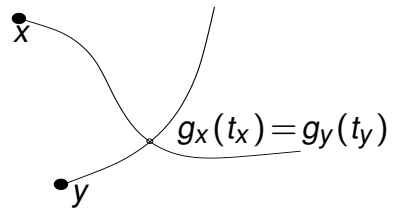

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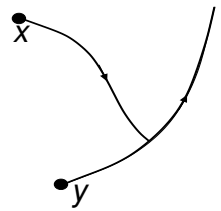

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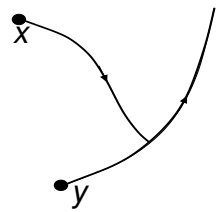

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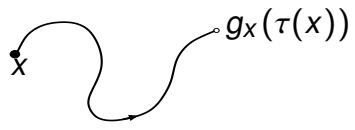

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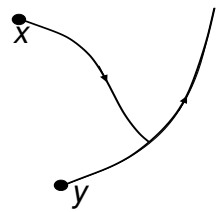

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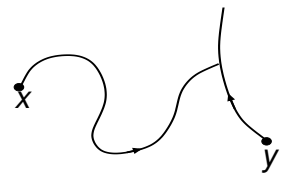

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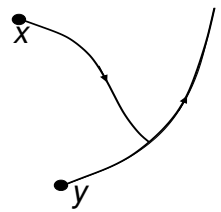

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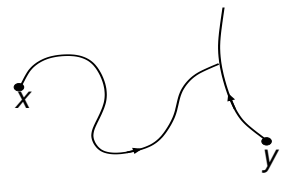

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Assume that

$$\forall t \geq 0,\ \mathbb{E}\left[\sup_{s\in[0,t]} \|h(s)\|^2\right] < \infty \qquad (\star)$$

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Corollary: (i) The Brownian model is well defined.
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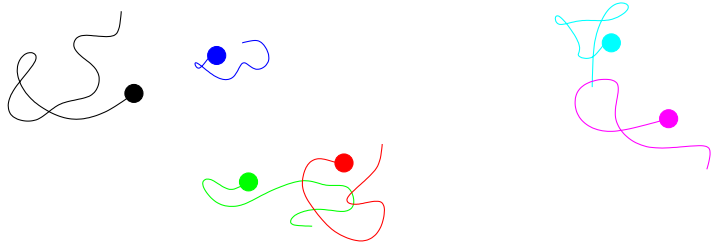
paths on $[0, t]$ 

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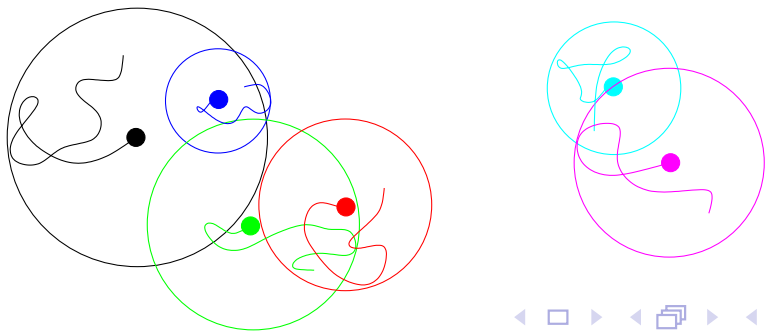

Plan

1 Introduction

2 Existence result

3 Percolation result

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The Loop property at scale $\alpha > 0$ holds if there exists an event $\mathcal{L}(\alpha)$ and a function $0 < r(\alpha) < \alpha$ s.t.:

- $\mathbb{P}[\bar{\xi} \in \mathcal{L}(\alpha)] > 0$.
- $\mathcal{L}(\alpha)$ is a local event: $\bar{\xi} \in \mathcal{L}(\alpha) \Leftrightarrow \bar{\xi}_{|B(\alpha)} \in \mathcal{L}(\alpha)$.
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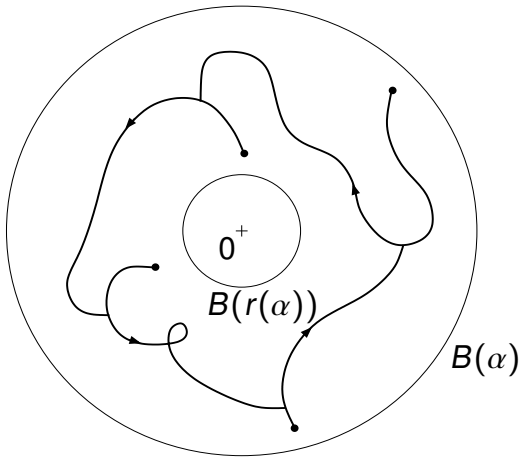

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The model defined on $\bar{\xi}$ does not percolate if the union of stopped grains

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contains no unbounded connected components.

Theorem 2: [Coupier, Dereudre, Gouéré '25]
Assume that the model is a.s. well defined on $\bar{\xi}$ and let $h \sim \mu$. Assume also that

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(*ii*) The Line segment model does not percolate a.s. if $\mathbb{E}[V^2] < \infty$, where $h(t) = tV \times U$.

Percolation result

The model defined on $\bar{\xi}$ does not percolate if the union of stopped grains

$$\bigcup_{x\in\xi} G_x(\tau(x))$$

contains no unbounded connected components.

Theorem 2: [Coupier, Dereudre, Gouéré '25]
Assume that the model is a.s. well defined on $\bar{\xi}$ and let $h \sim \mu$. Assume also that

$$\exists t > 0 \text{ s.t. } \mathbb{E}\left[\sup_{s\in[0,t]} \|h(s)\|^2\right] < \infty \qquad (\star\star)$$

and that the Loop property $\mathcal{L}(\alpha)$ holds at any (small) scale $\alpha > 0$. Then the model defined on $\bar{\xi}$ does not percolate a.s.

Corollary: (i) The Brownian model does not percolate a.s.
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Reduction to 'No infinite Forward sets'

We associate to the configurations of stopped grains, the directed graph $\overrightarrow{\mathrm{Graph}}(\bar{\xi})$ defined by:

- Vertex set: ξ.
- Edge set: $x \to y \Leftrightarrow$ the grain x is stopped by y.
  Let $s(x) := y$ the successor of x.

Rmk: $\overrightarrow{\mathrm{Graph}}(\bar{\xi})$ is outdegree-one.

Forward set of x: $\mathrm{For}(x) := \{y \in \xi : x \to \ldots \to y\}$.

Backward set of x: $\mathrm{Back}(x) := \{y \in \xi : y \to \ldots \to x\}$.

- Sufficient to prove that a.s. $\forall x \in \xi$, $\mathrm{Card}(\mathrm{For}(x)) < \infty$.
  I.e. a.s. $\forall x \in \xi$, $\mathrm{For}(x)$ contains a loop.

  (by mass transport principle + soft arguments)

Absurd: we assume that $\mathbb{P}[\exists x \in \xi,\ \mathrm{Card}(\mathrm{For}(x)) = \infty] > 0$.

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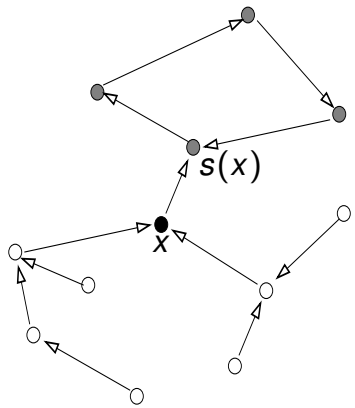

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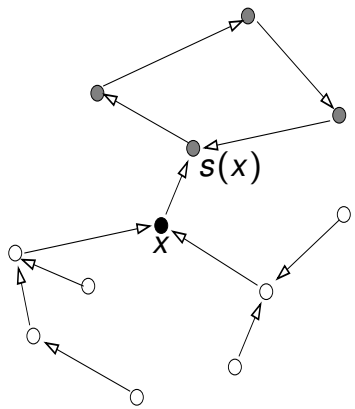

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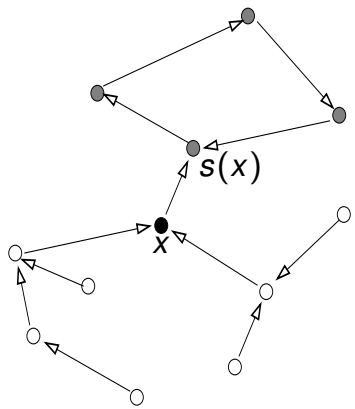

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Reduction to 'No infinite Forward sets of α-bad points'

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Let $\alpha > 0$.

An α-good point is an opportunity to stop an (infinite) forward set.

Definition: $x \in \xi$ is an α-good point if

we can place a (small) loop inside $B(\cdot, \alpha)$,
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And α-bad point if not α-good point.

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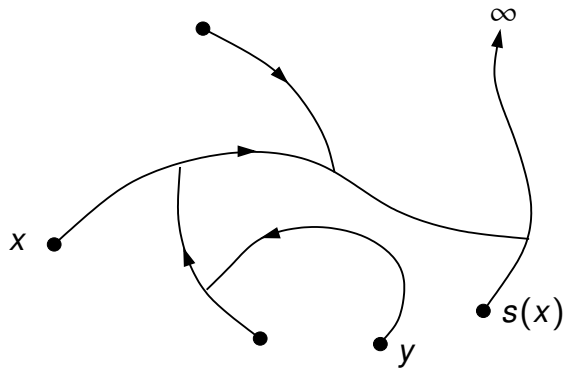

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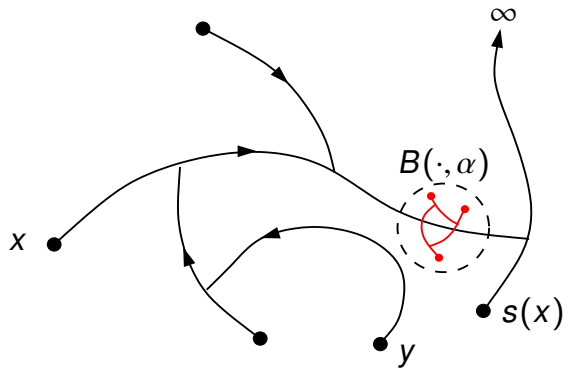

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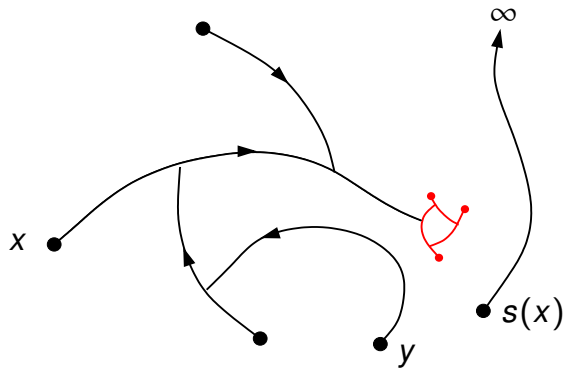

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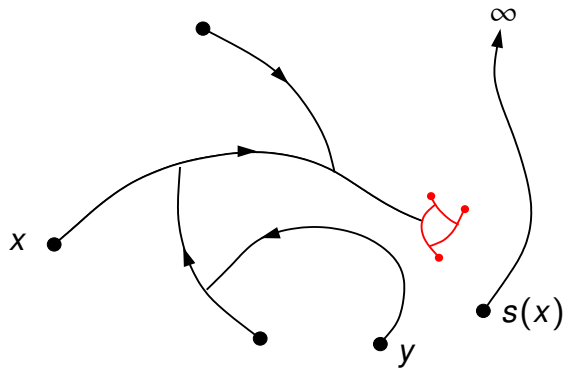

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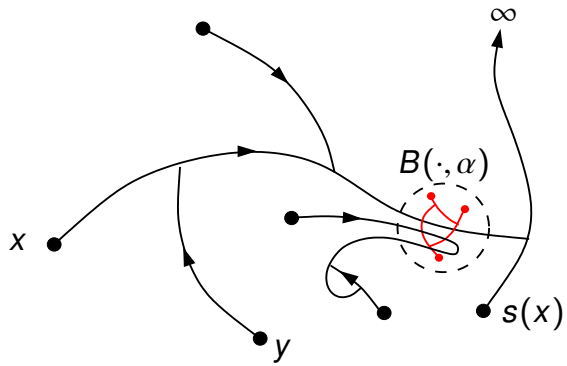

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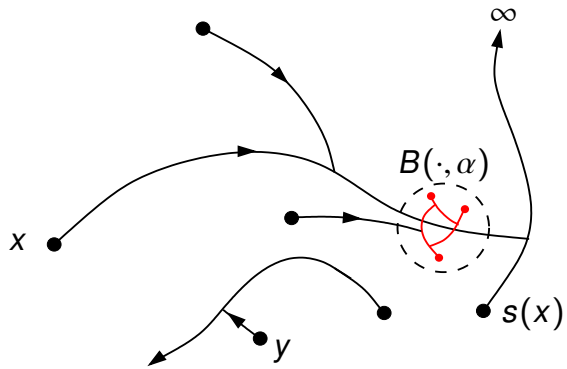

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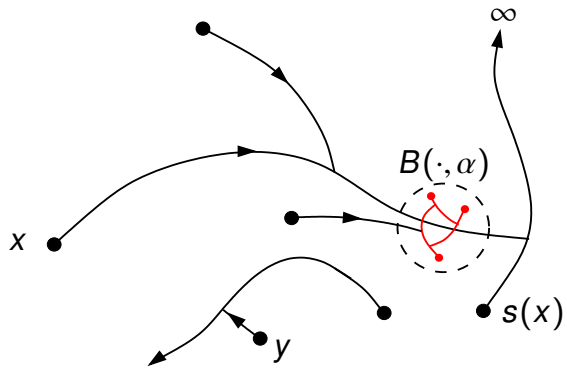

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An augmented #-model

Problem: α-goodness needs non-local informations as $\tau(x)$, $s(x)$.

Goal: witness α-goodness in a more local way.

We use an augmented #-model in which each grain is stopped later.
I.e. $\tau^{\#}(x) \geq \tau(x)$; the #-grains are bigger.

Advantage: We gain locality.

Thanks to $(\star\star)$, the #-model is local.

Disadvantage: We lose information.

The #-grains around x determine what really happens around x **up to a finite number of scenarios**.

$x \in \xi$ is $(\alpha, \#)$-good if it is α-good in each of these scenarios.

Conclusion: We prove $\mathbb{P}^0[0 \text{ is } (\alpha, \#)\text{-bad}] \to 0$ as $\alpha \to 0$

$+$ locality of the #-model

$\Rightarrow$ no percolation of the $(\alpha, \#)$-bad points. (by a multiscale argument)

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