Rigidity of long-range Gibbs point processes

Rafaël Digneaux

Advisor: David Dereudre

University of Lille

GeoSto26 Rouen

- **Point configuration**: a locally finite subset $\gamma \subset \mathbb{R}^d$
- **Point process**: random γ, or a measure $\mu(\mathrm{d}\gamma)$ on Ω

Some examples:

(1) If f is random polynomial/holomorphic function:

$$\gamma = \{\text{zeros of } f\}.$$

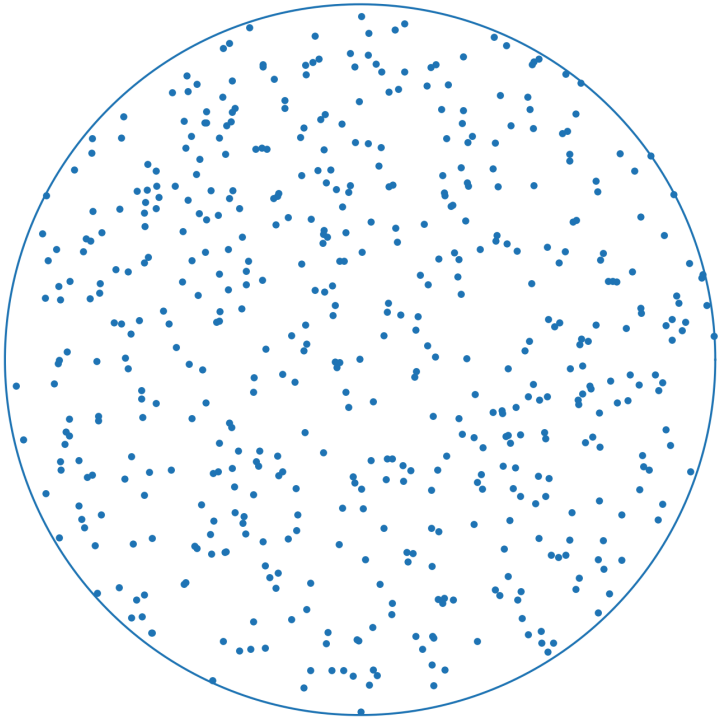
(2) If M is random matrix:

$$\gamma = \{\text{eigenvalues of } M\}.$$

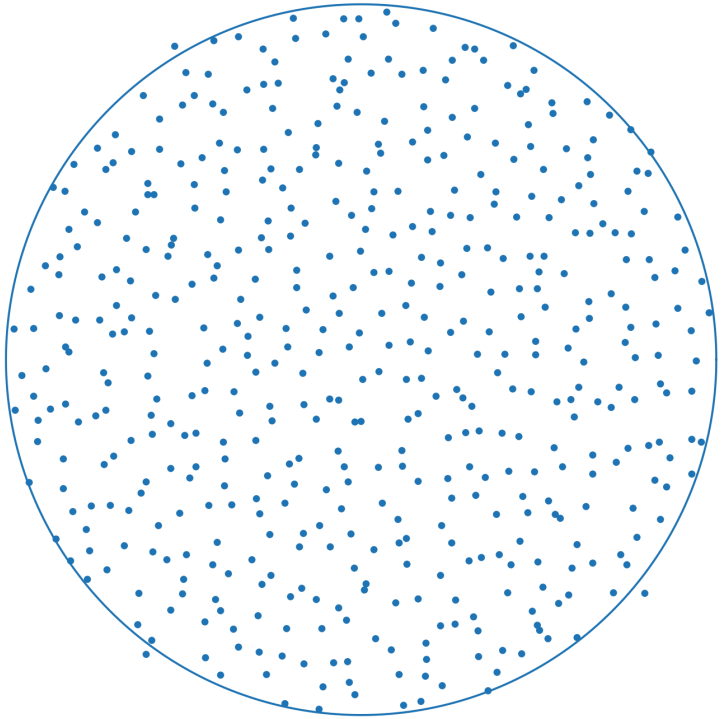
(3) If $H : \Omega \to \mathbb{R}$ is an energy, the Gibbs measure associated.

Here the interaction is **repulsive** and **long-range**.

Poisson



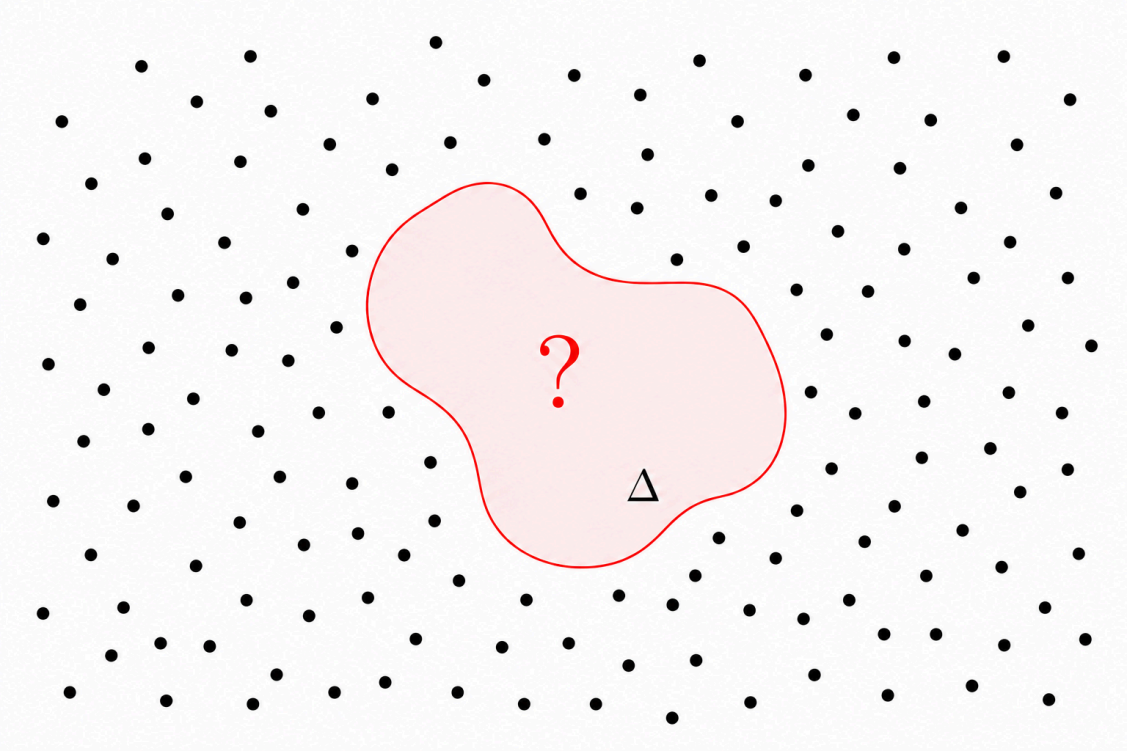
Random matrix



Goal: studying **rigidity** properties for **infinite-volume Gibbs point processes**

Number-Rigidity: $\forall \Delta$, number of points $N_\Delta(\gamma) = F(\gamma_{\Delta^c})$ a.s.

[*Aizenman-Martin, 1980*] [*Peres-Sly, 2014*]



Setting:
(1) space $(\Omega, \mathcal{F}, \mu_{\text{ref}}(\mathrm{d}\gamma))$,
(2) functional $H : \Omega \to \mathbb{R} \cup \{+\infty\}$,
(3) temperature $T > 0$.

The Gibbs measure is

$$\mu_T(\mathrm{d}\gamma) = \frac{e^{-H(\gamma)/T}}{Z_T} \mu_{\text{ref}}(\mathrm{d}\gamma).$$

As $T \to +\infty$, then $\mu_T \to \mu_{\text{ref}}$ (*disorder*).

As $T \to 0^+$, then μ_T concentrates on minimizers of H (*order*).

$\to$ Phases transitions!

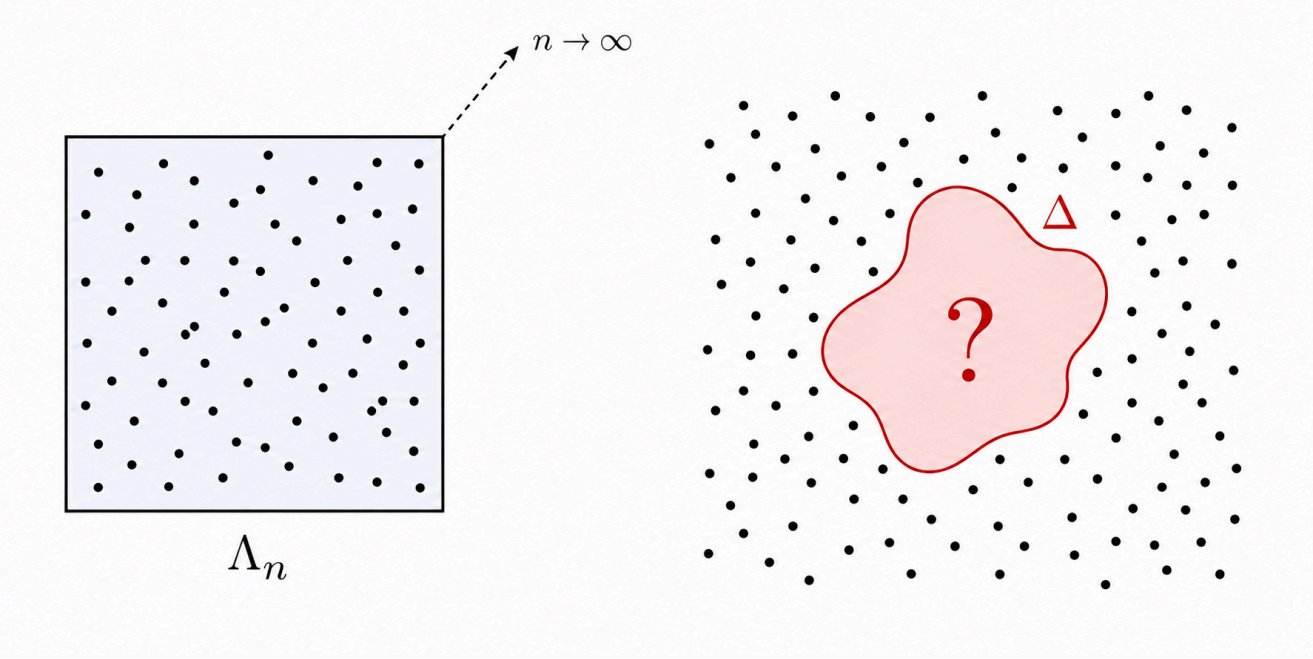
Problem: in **infinite volume**, energy H usually does not exist

(1) How to define the model ?

→ **thermodynamic limit**: finite volume Λ_n with $n \to +\infty$

(2) What property should it satisfy ?

→ **DLR equations**: conditional law of γ_Δ given γ_{Δ^c}



→ Existence or uniqueness are difficult.

In finite volume Λ_n, with $\gamma = \{x_1, ..., x_n\}$,

$$H(x_1, ..., x_n) = \sum_{i<j\leq n} V(x_i, x_j) + \sum_i V_{\text{ext}}(x_i),$$

- V_{ext} is confining potential
- V is a **pairwise interaction potential**
- $V(x, y) = V(\|x - y\|)$ (*isotropic, stationary*)
- V decreasing (*repulsive*)
- V not integrable at infinity (*long-range*)

Some choices of potential V:

(1) Log-interaction:

$$V(x,y) = -\log\|x-y\|, \quad d = 1, 2.$$

(2) 3D Coulomb interaction:

$$V(x,y) = \frac{1}{\|x-y\|}, \quad d = 3.$$

(3) **s-Riesz interactions**:

$$V_s(x,y) = \begin{cases} \|x-y\|^{-s} & \text{if } s > 0 \\ -\log\|x-y\| & \text{if } s = 0 \\ -\|x-y\|^{-s} & \text{if } s < 0 \end{cases}$$

Coulomb case: $s = d - 2$.

Conjecture: s-Riesz gas is Number-Rigid iff $s \leq 0$.

[*Dereudre-Digneaux, 2025, to appear in J. Appl. Prob.*]

Theorem [DD25]

If γ is a one-dimensional stationary point process with finite L^1 transport cost from Lebesgue, then γ is **Number-Rigid**.

Application:

Theorem [DD25]

1D Riesz gases for $s \in (-2, -1]$ are **Number-Rigid**.

This includes $s = -1$ (1D Coulomb)!

Some results:

(1) 1D log-gas **NR**, **DLR** [*Dereudre-Hardy-Leblé-Maïda, 2019*]

(2) Riesz $s \in (d-1, d)$ **not NR**, **DLR** [*Dereudre-Vasseur, 2021*]

(3) $d \geq 3$ Coulomb **not NR** [*Thoma, 2023*]

(4) 2D Coulomb **NR**, **DLR** [*Leblé, 2025*]

$\rightarrow$ Link between DLR and NR ?

$\rightarrow$ How about $s \in (d-2, d-1)$?

(V) NR and DLR equations

[*Dereudre-Digneaux-Henry, work in progress*]

Theorem

General potential V

(1) **DLR** equations implies **non NR** when V is bounded at infinity.
(2) **DLR** equations implies **NR** when $V(r) \underset{r\to+\infty}{\longrightarrow} -\infty$.

Literature: Riesz gases $s \in (d-1, d)$ and at $s = d-2$ are **not NR**.

Theorem

The Riesz gases for $s \in ((d-2)_+, d)$ are **not Number-Rigid.**

[*Dereudre-Digneaux-Henry, work in progress*]

How to make sense of $H_\Delta(\eta_\Delta|\gamma_{\Delta^c})$?

It should be

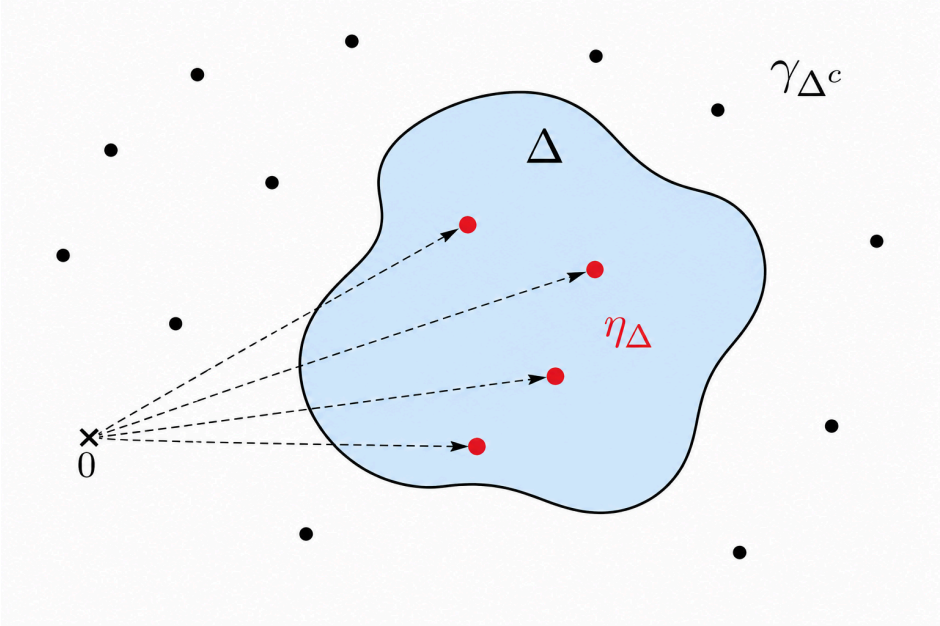
$$H_\Delta(\eta_\Delta|\gamma_{\Delta^c}) =? \sum_{x\in\eta_\Delta}\sum_{y\in\gamma_{\Delta^c}} V(y-x)$$

Does not exist when V is not integrable (*long-range*).

[*Dereudre-Digneaux-Henry, work in progress*]

Idea in the literature "Move functions":

$$H_\Delta(\eta_\Delta|\gamma_{\Delta^c}) =? \sum_{x\in\eta_\Delta}\sum_{y\in\gamma_{\Delta^c}} [V(y-x)-V(y)].$$

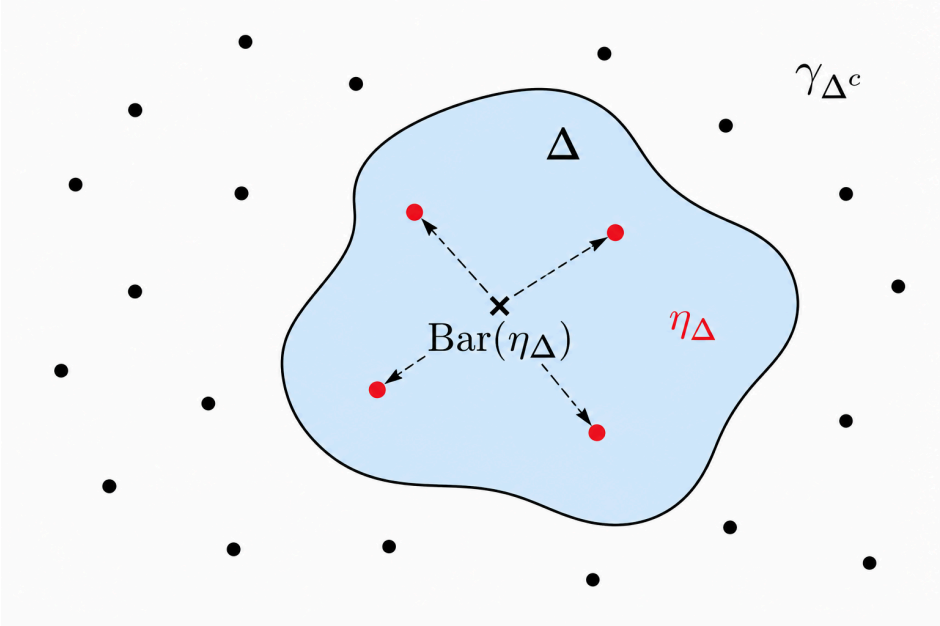


This removes the first-order diverging term.

[*Dereudre-Digneaux-Henry, work in progress*]

Further idea:

$$H_\Delta(\eta_\Delta|\gamma_{\Delta^c}) =? \sum_{x\in\eta_\Delta}\sum_{y\in\gamma_{\Delta^c}}[V(y-x)-V(y-\mathrm{Bar}(\eta_\Delta))].$$



This removes the two-order diverging terms.

References

[AM80] M. Aizenman and P. A. Martin, "Structure of Gibbs states of one-dimensional Coulomb systems," 1980.

[PS14] Y. Peres and A. Sly, "Rigidity and tolerance for perturbed lattices." 2014.

[Der+19] D. Dereudre, A. Hardy, T. Leblé, and M. Maïda, "DLR equations and rigidity for the Sine-beta process." 2019.

[DV21] D. Dereudre and T. Vasseur, "Number-Rigidity and beta-Circular Riesz gas." 2021.

[Tho23] E. Thoma, "Non-rigidity Properties of the Coulomb Gas." 2023.

[Leb24] T. Leblé, "DLR equations, number-rigidity and translation-invariance for infinite-volume limit points of the 2DOCP." 2024.

[DD25] D. Dereudre and R. Digneaux, "Rigidity of one-dimensional point processes via optimal transport." 2025.