Scaling limit of non-simple Conformal Loop Ensembles

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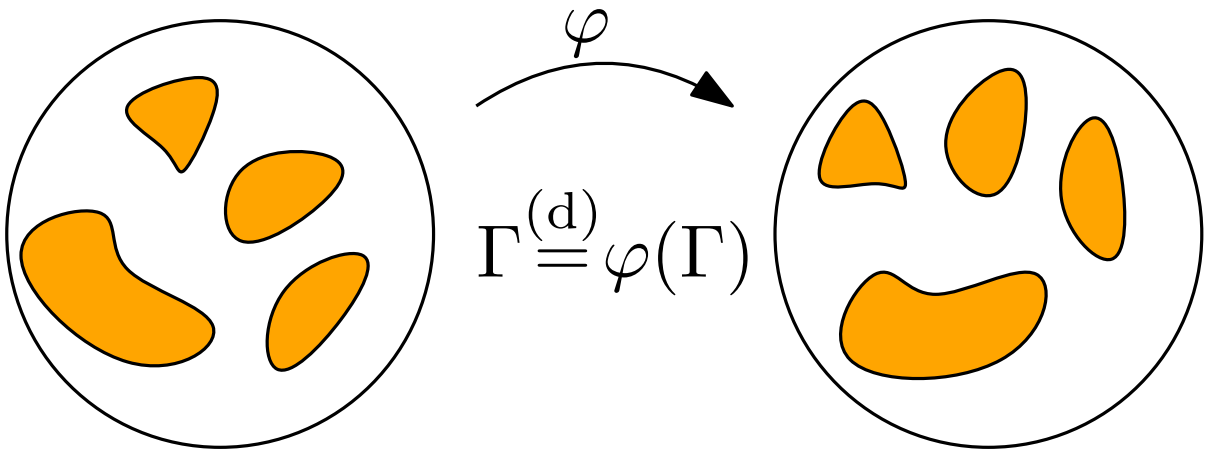
University of Cambridge

Joint work with Konstantinos Kavvadias, Jason Miller and Yi Tian

Conformal loop ensemble (CLE) on the unit disk $\mathbb{D}$

collection Γ of random loops that do not cross each other

- Invariant in law under Möbius transformations



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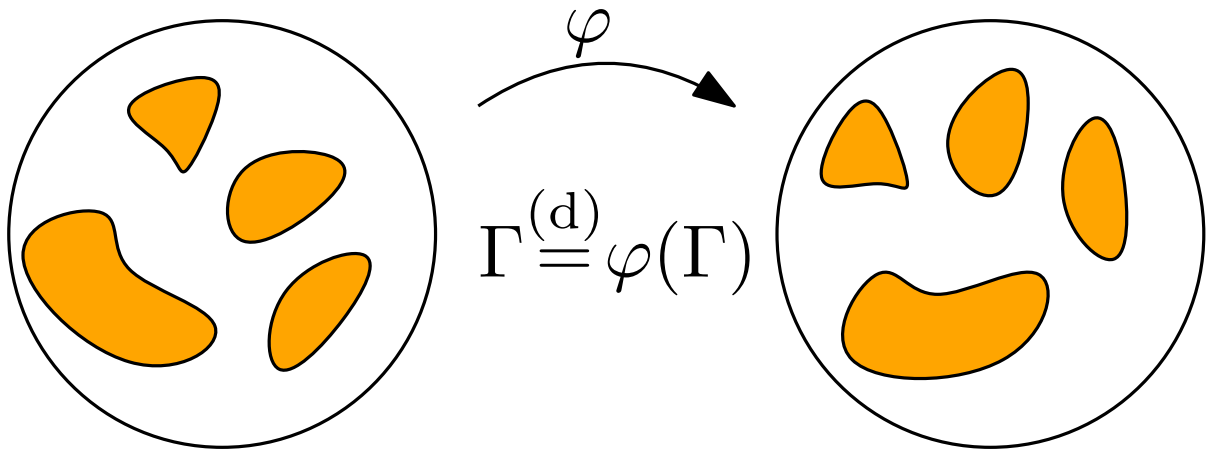
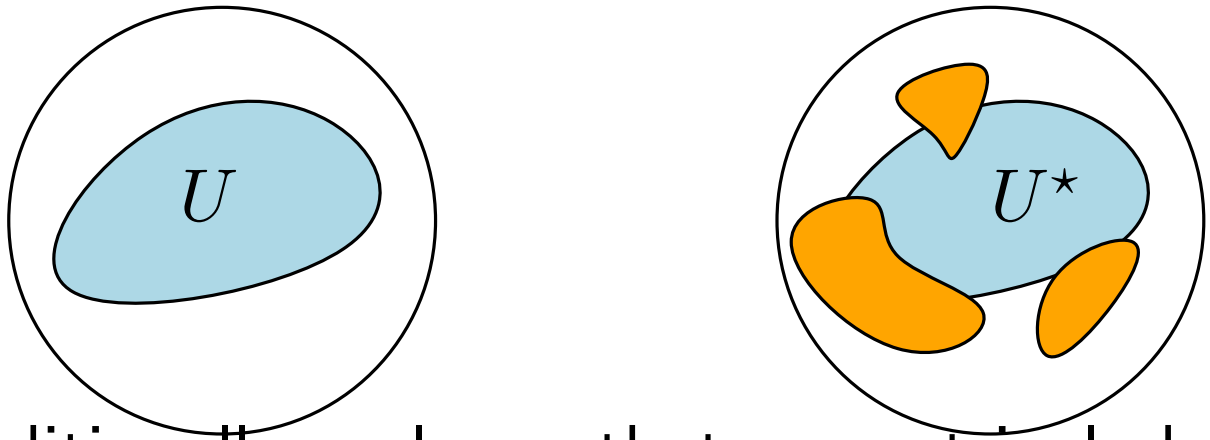


- Domain Markov property: $\forall$ simply connected open $U \subset \mathbb{D}$,

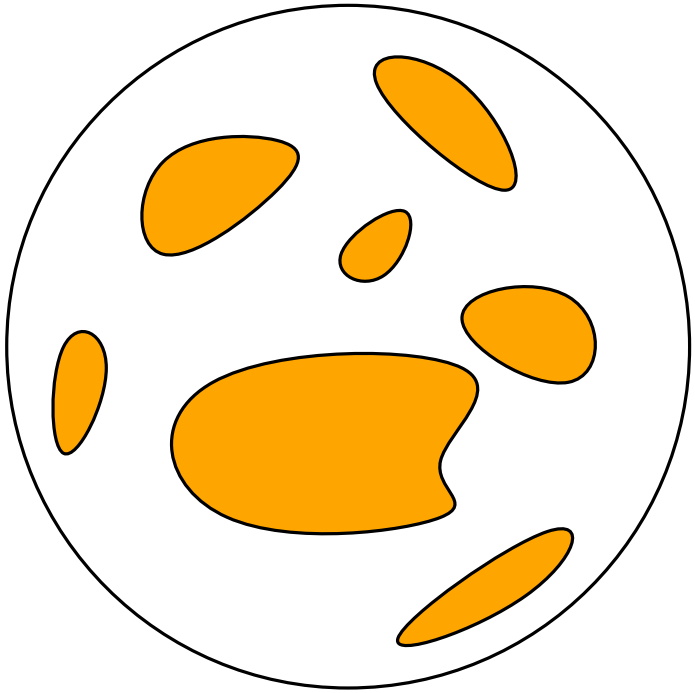


Conditionally on loops that are not included in U,
the loops $U^\star$ form independent CLEs in components of $U^\star$.

Conformal loop ensembles (CLE) on the unit disk $\mathbb{D}$

There exists a family CLE_κ for $\kappa \in (8/3, 8)$ satisfying the axioms.
Constructed by Sheffield using branching $\mathrm{SLE}_\kappa(\kappa - 6)$ curves.

For $\kappa \leq 4$:



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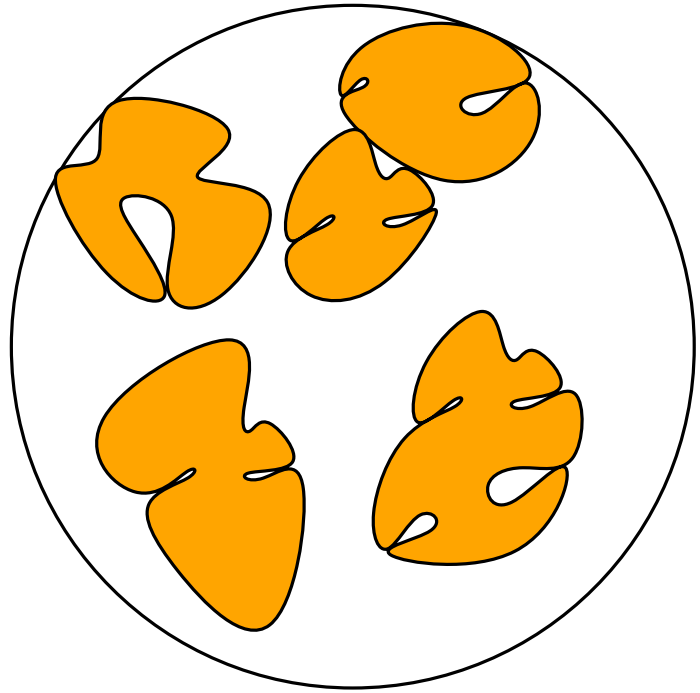

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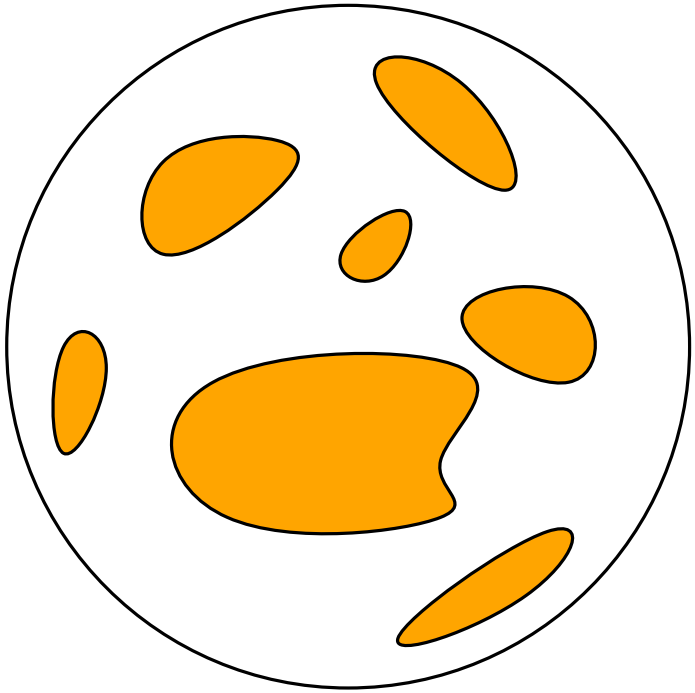

Simple loops
Do not touch each other
Do not touch $\partial\mathbb{D}$

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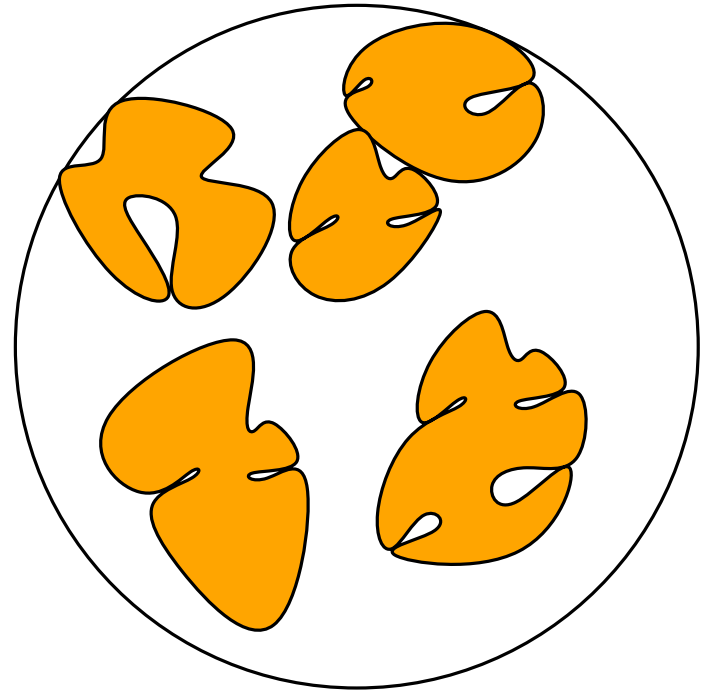

Non-simple loops: self-intersect
Can touch each other
Can touch $\partial\mathbb{D}$

Conformal loop ensembles (CLE) on the unit disk $\mathbb{D}$

Conjectural scaling limit of cluster interfaces in critical statistical mechanics models on a 2D lattice

- Percolation ($\kappa = 6$): by Smirnov ('01) for the triangular lattice
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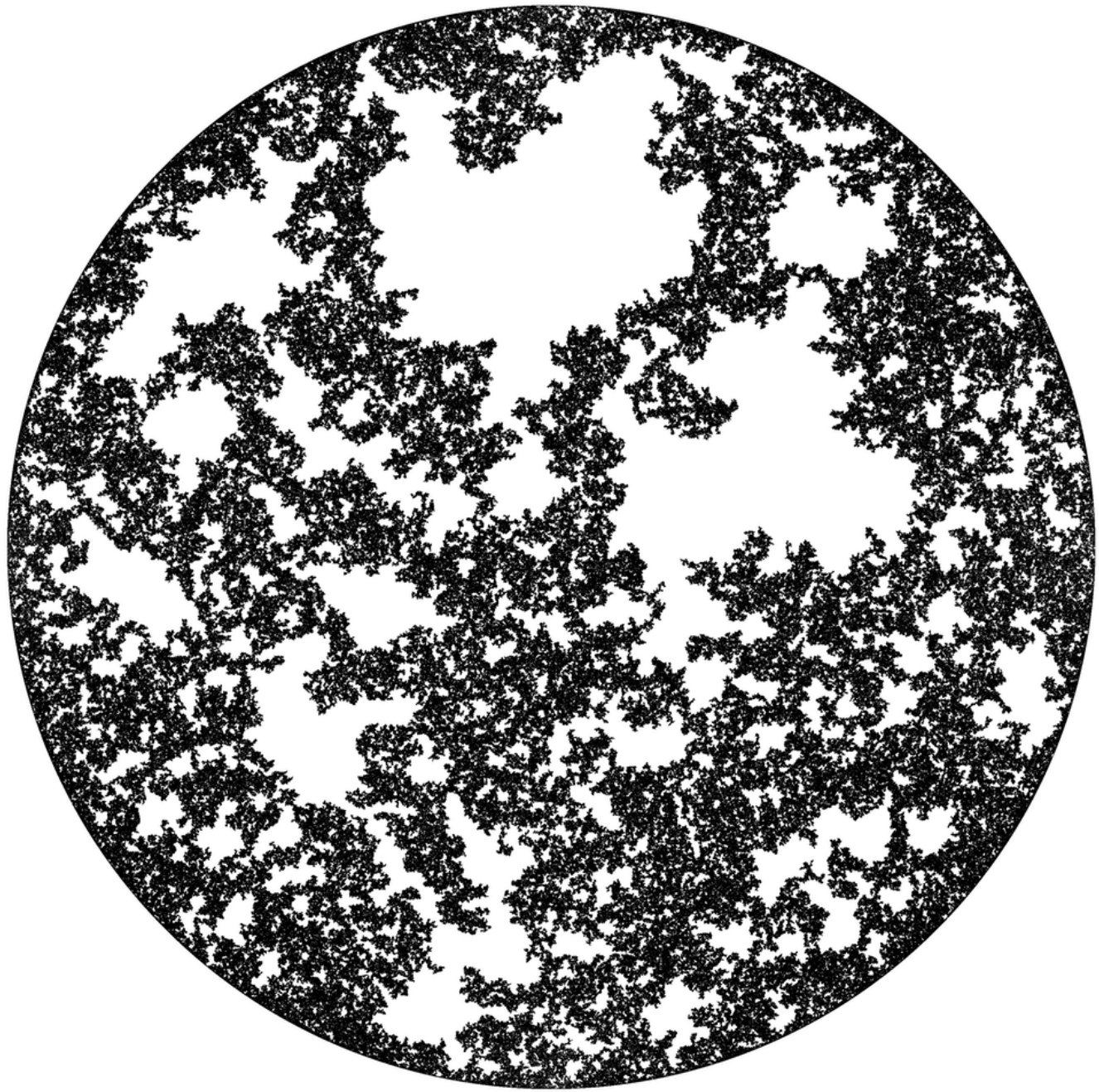
- Random cluster model
- $O(n)$ loop model

  $\vdots$

Similar scaling limits are expected for random lattices, i.e. random planar maps.

Simulation of a CLE_4 in $\mathbb{D}$



Graph distance on non-simple CLEs

For $\kappa \in (4, 8)$, let Γ^κ be a CLE_κ in $\mathbb{D}$.

Say that two loops are adjacent if they touch each other.

$D^\kappa =$ graph distance between loops of Γ^κ.

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For all $x \in \mathbb{D} \cap \mathbb{Q}^2$, let $\mathscr{L}^\kappa(x)$ be the loop surrounding x

By conformal invariance and the Markov property, $D^\kappa(\partial\mathbb{D}, \mathscr{L}^\kappa(x))$ is a geometric r.v. with parameter

$$1/\mathfrak{a}_\kappa = \mathbb{P}\left(\mathscr{L}^\kappa(x) \cap \partial\mathbb{D} \neq \emptyset\right)$$

Scaling limit of non simple CLE

Theorem (KKMT '26)

For the product topology,

$$\left(\frac{1}{\mathfrak{a}_\kappa} D^\kappa(\mathscr{L}^\kappa(x), \mathscr{L}^\kappa(y)\right)_{x,y\in\mathbb{D}\cap\mathbb{Q}^2} \xrightarrow[\kappa\downarrow 4]{(\mathrm{d})} (D(\mathscr{L}(x), \mathscr{L}(y))_{x,y\in\mathbb{D}\cap\mathbb{Q}^2}$$

where D is a geodesic metric on a CLE_4 Γ such that:

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where D is a geodesic metric on a CLE_4 Γ such that:

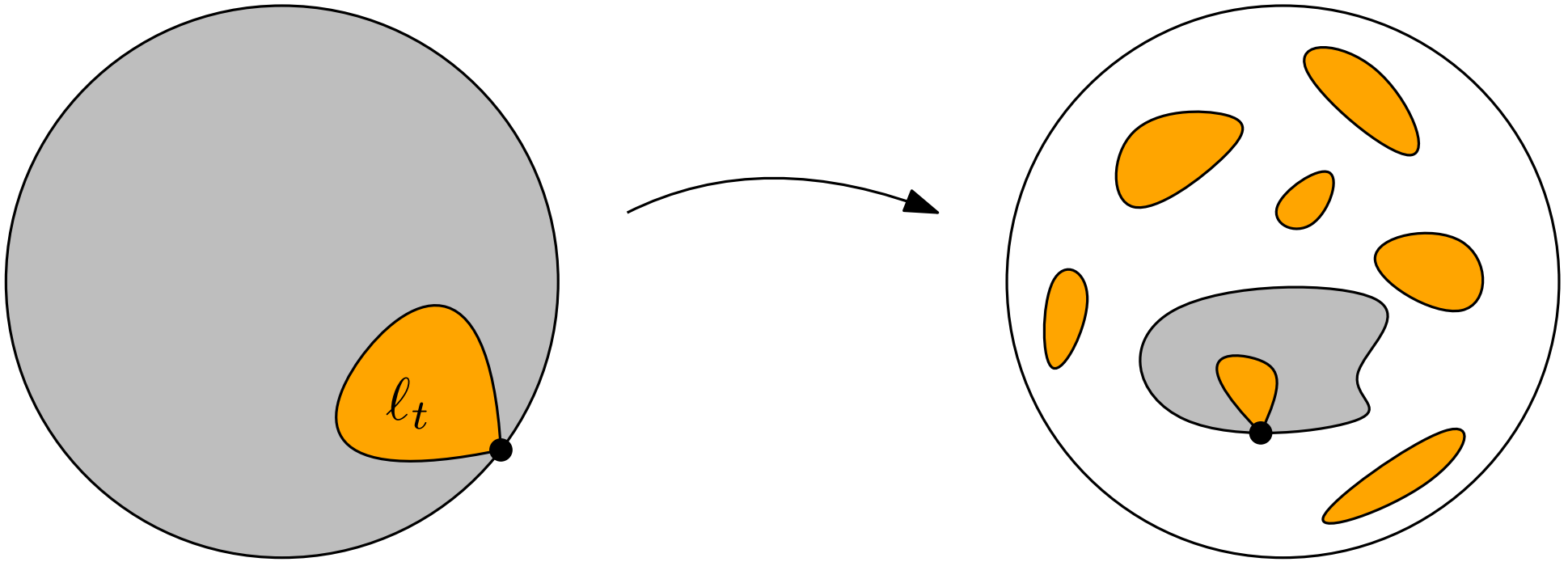
- D is a measurable function of Γ
- D is conformally invariant: for all conformal map φ on $\mathbb{D}$, the CLE_4 metric on $\varphi(\Gamma)$ is the pushforward of D by φ.
- D is local: for all simply connected $U \subset \mathbb{D}$, the metrics induced by D on components of $U^\star$ are functions of the restrictions of Γ

We also show the convergence of distances to $\partial\mathbb{D}$ or arcs of $\partial\mathbb{D}$

Scaling limit of non simple CLE

Theorem (KKMT '26)

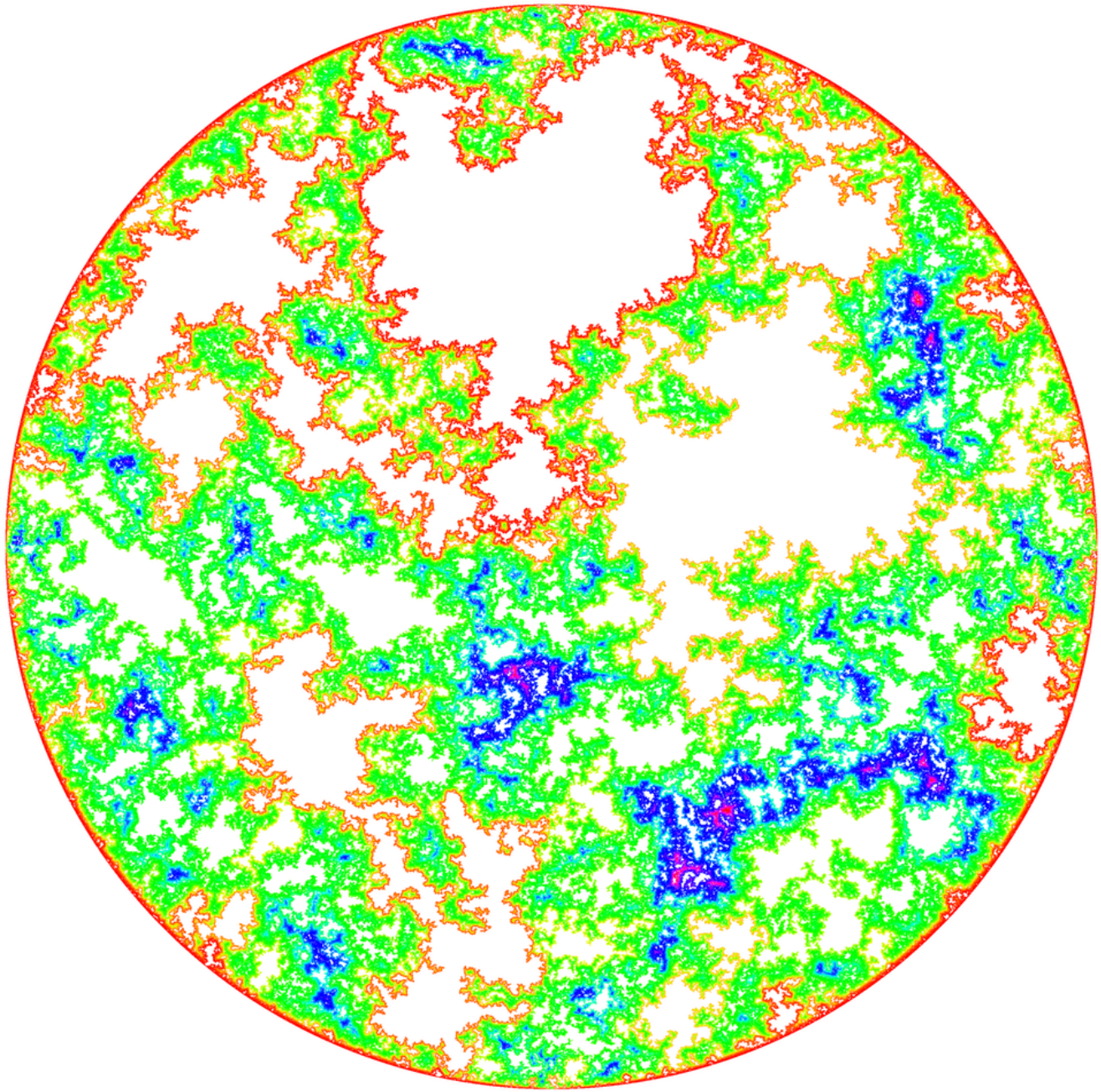
Moreover, the distances to the boundary are described by the uniform exploration introduced by Werner and Wu ('13).



PPP of SLE_4 bubbles $(\ell_t)_{t\geq 0}$ loop discovered at time t

We also show that D is characterised by the above properties.

Simulation of the distances to the boundary



by Yi Tian

Relationships and differences with other works

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Our approach is inspired by two different settings:

- Scaling limit of random maps (Le Gall '13, Miermont '13...):
  - No natural embedding in the plane
  - Existence of subsequential limits for a strong topology, e.g. Gromov-Hausdorff. $\Longrightarrow$ existence of geodesics.
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- Construction of conformally covariant metrics,
  e.g. the Liouville quantum gravity metric (Ding, Dubédat...)
  - Natural embedding in the plane and conformal covariance $\Longrightarrow$ possible to compare with the Euclidean metric
  - Tightness for the topology of uniform convergence (via independence accross scales) $\Longrightarrow$ $\exists$ geodesics
  - Uniqueness via bi-Lipschitz equilvalence and Efron-Stein

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In our case:

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Main steps of the proof

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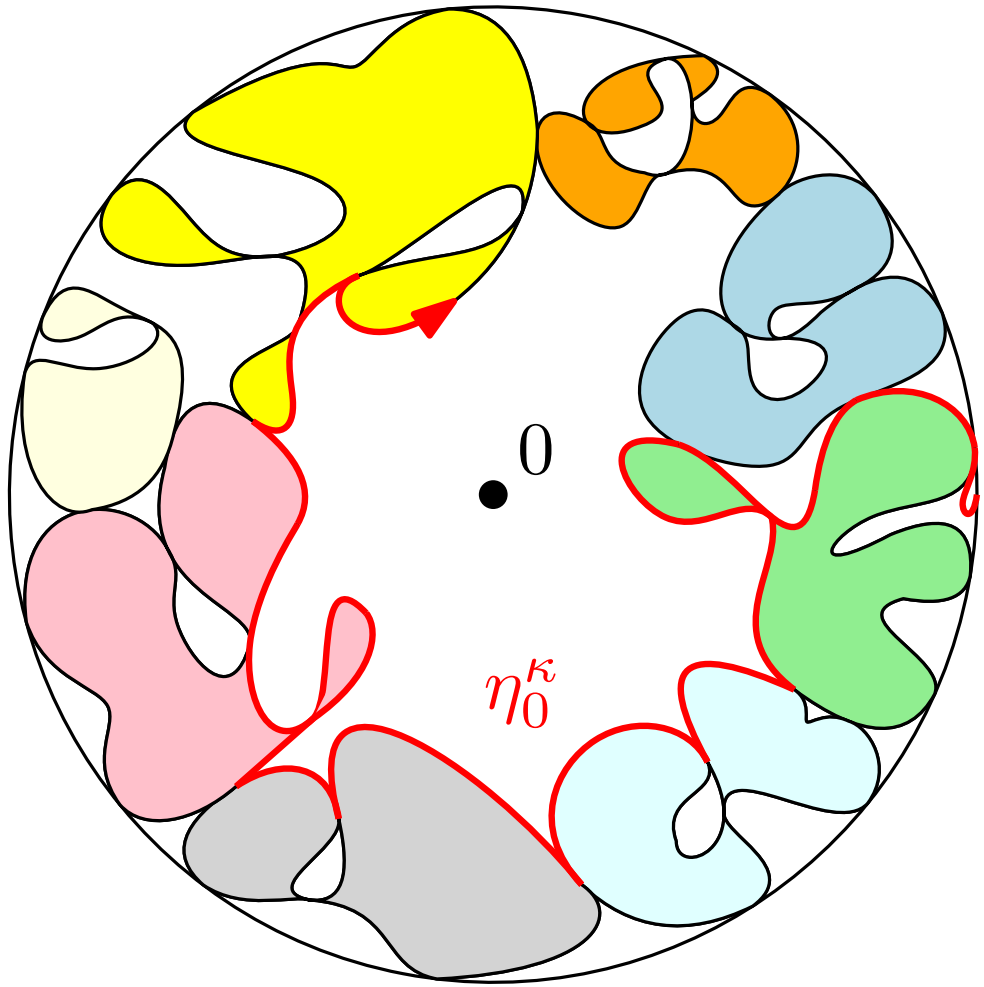
- Existence of geodesics in any metric satisfying some axioms
- Uniqueness:
  - Geodesics can be approximated by going up and down in the tree of geodesics to the root.
  - The tree of geodesics to the root is a.s. determined by the uniform exploration
  - The uniform exploration is a.s. determined by the CLE_4.

Existence of subsequential limits: uniform exploration

- Construction of the CLE_κ layer by layer via $\mathrm{SLE}_\kappa(\kappa-6)$



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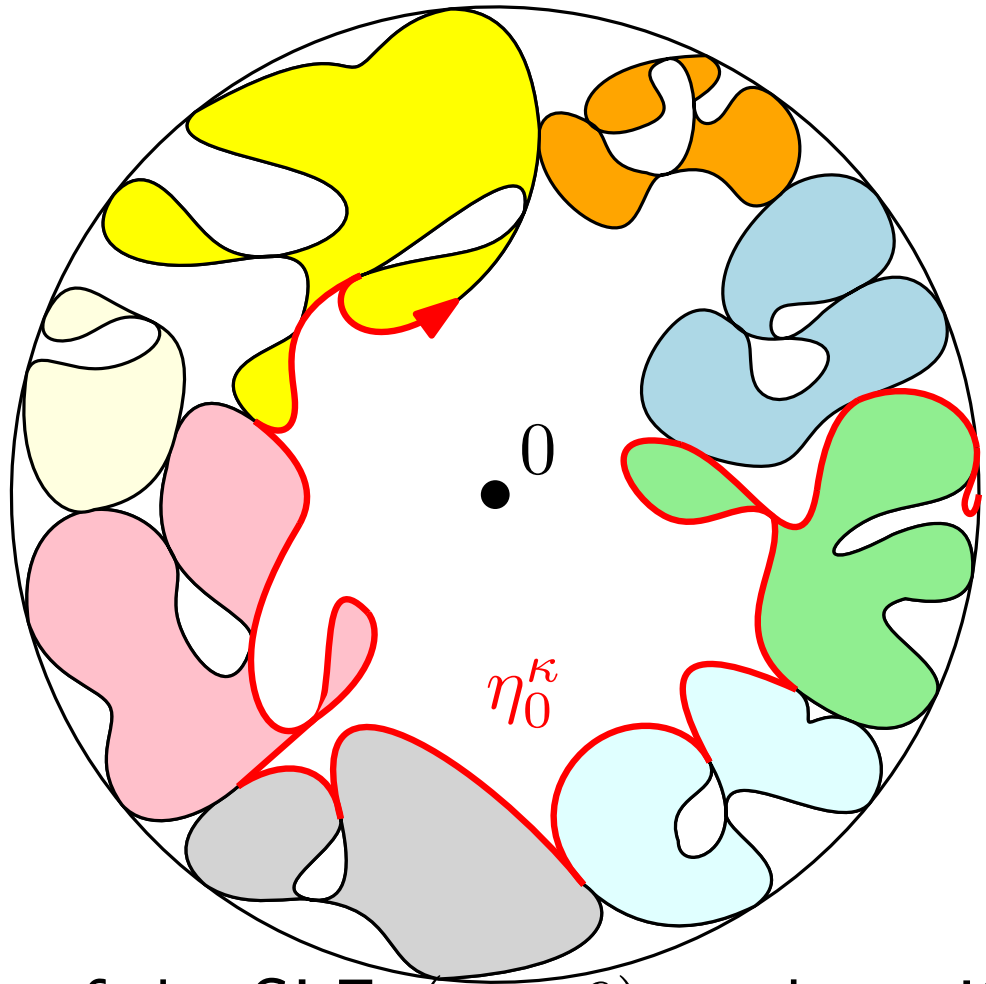
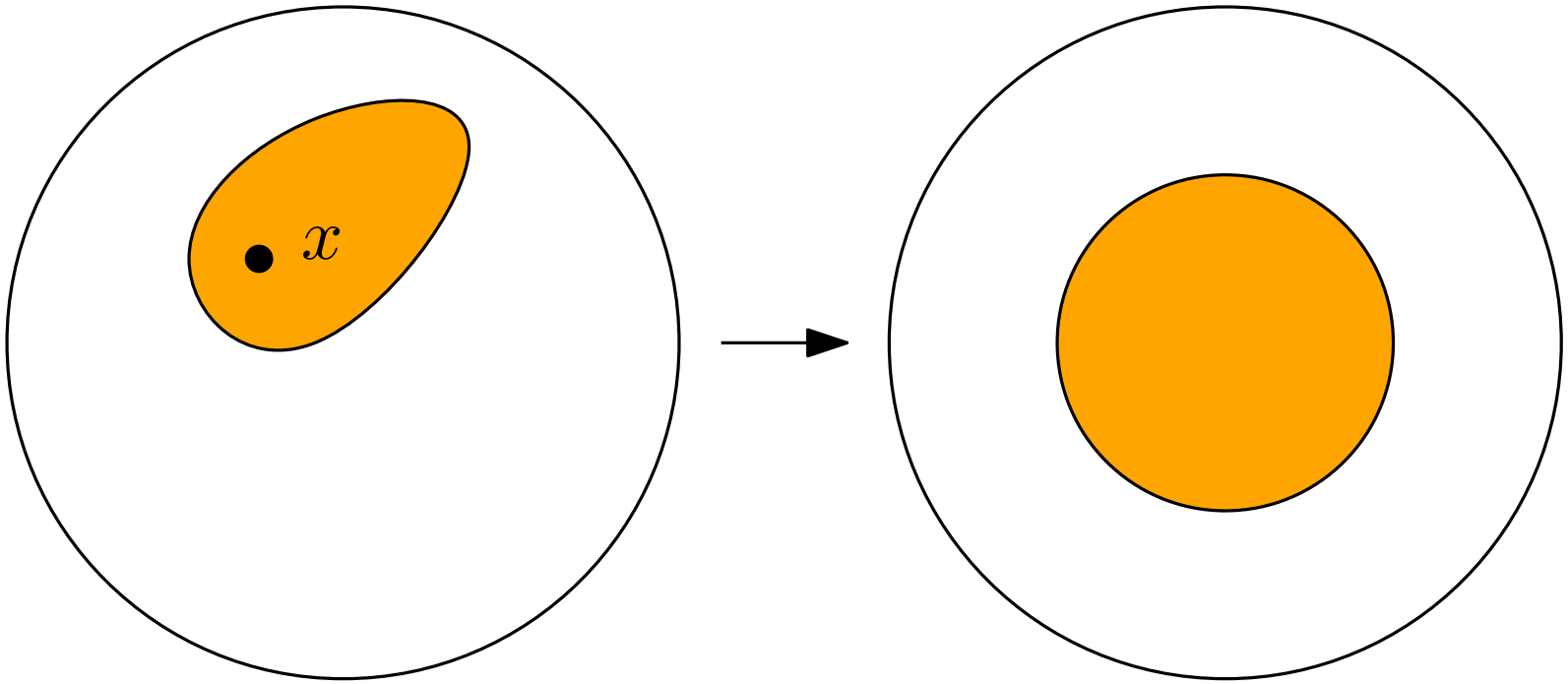


- Convergence of the $\mathrm{SLE}_\kappa(\kappa-6)$ to the uniform exploration (Aru, Holden, Powell and Sun '23)
- Times of completion of layers are renewal times.

Existence of subsequential limits: ball from a loop

- Take $x \in \mathbb{D} \cap \mathbb{Q}^2$. Conformally map to an annulus



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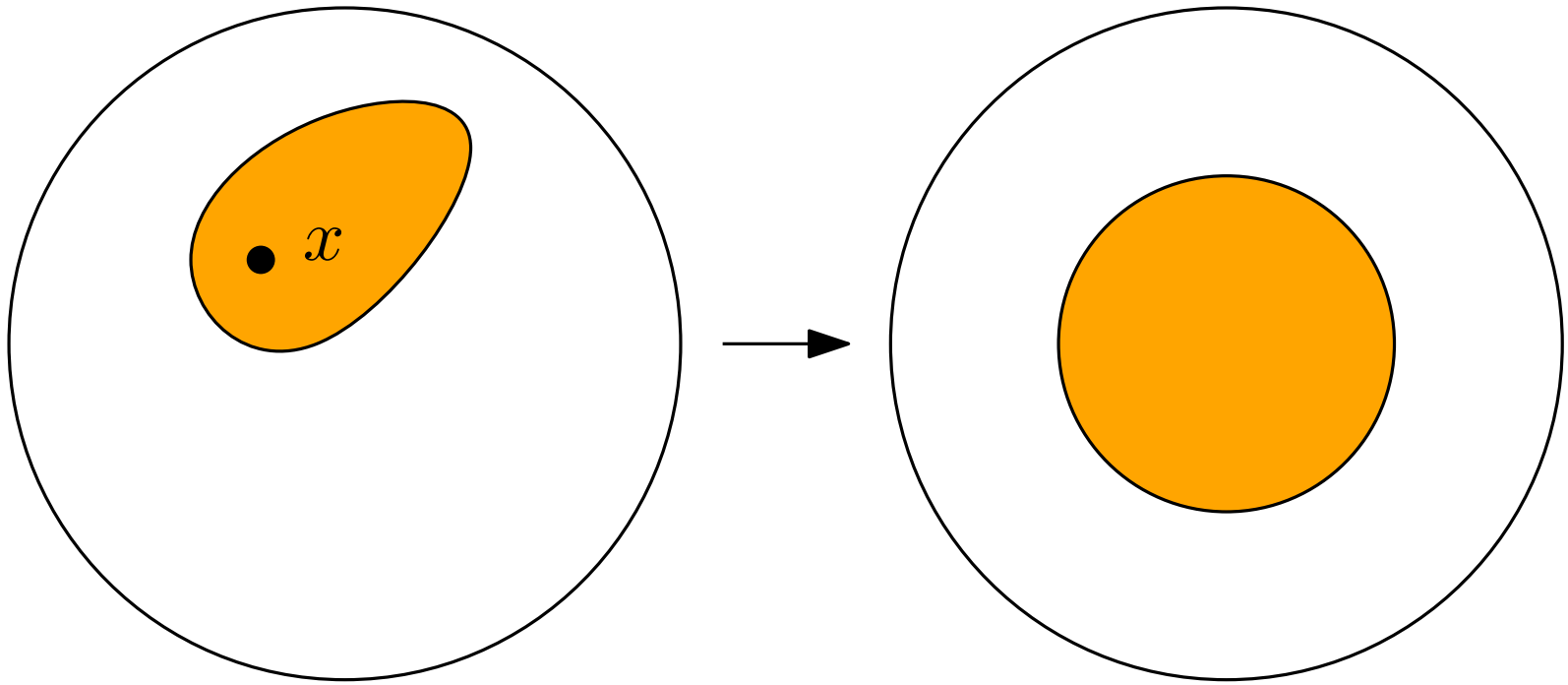

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- Apply inversion invariance

Sheffield, Watson, Wu ($\kappa = 4$) and Gwynne, Miller, Qiand ($\kappa > 4$)

This gives a distance on $\Gamma \cup \{\partial\mathbb{D}\}$.

After some work, we also get a distance without $\partial\mathbb{D}$.

Existence of subsequential limits: checking the axioms

- Conformal invariance: immediate by conformal invariance of the CLE_κ for $\kappa > 4$
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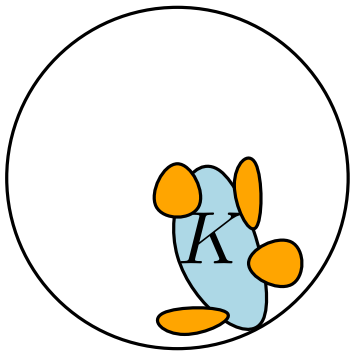
- Connection to the uniform exploration: already shown
- It remains to check locality (and some other axioms that are put under the carpet)

Main difficulty in checking locality by letting $\kappa \downarrow 4$:

Lemma Let $K \subset \overline{\mathbb{D}}$ be a connected compact set and F^κ be the closure of the union of K with the interiors of loops that intersect K. Then $F^\kappa \to F^4$ as $\kappa \downarrow 4$ in law.

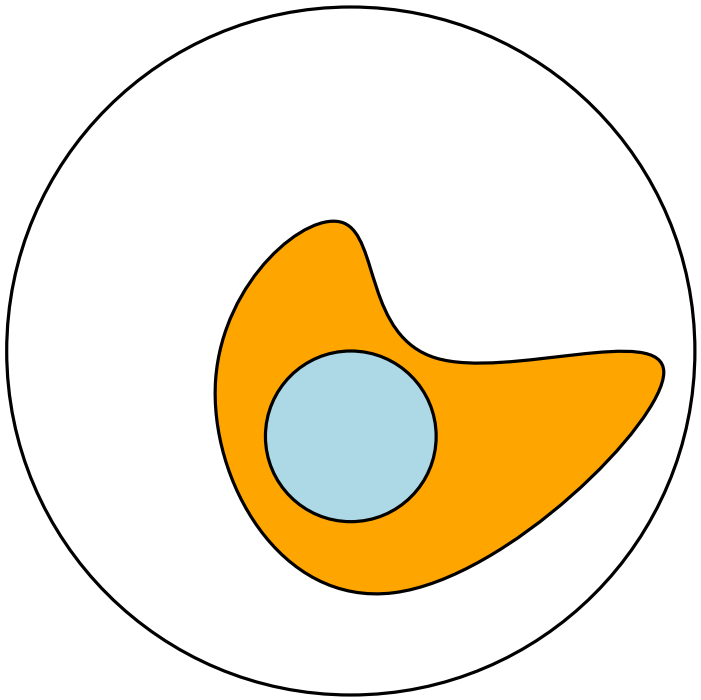


(not a consequence of the Hausdorff convergence of loops by Aru, Holden, Powell and Sun '23)

Existence of subsequential limits: checking the axioms

The previous lemma is a consequence of the following result:

Proposition (big non-simple CLE loops are fat)
Let $\kappa_n \downarrow 4$. For all $\varepsilon, \varepsilon' > 0$, there exists $\delta > 0$ such that for all n, with probability at least $1 - \varepsilon'$, all the loops of Γ^{κ_n} with diameter at least ε surround a Euclidean ball of radius δ.



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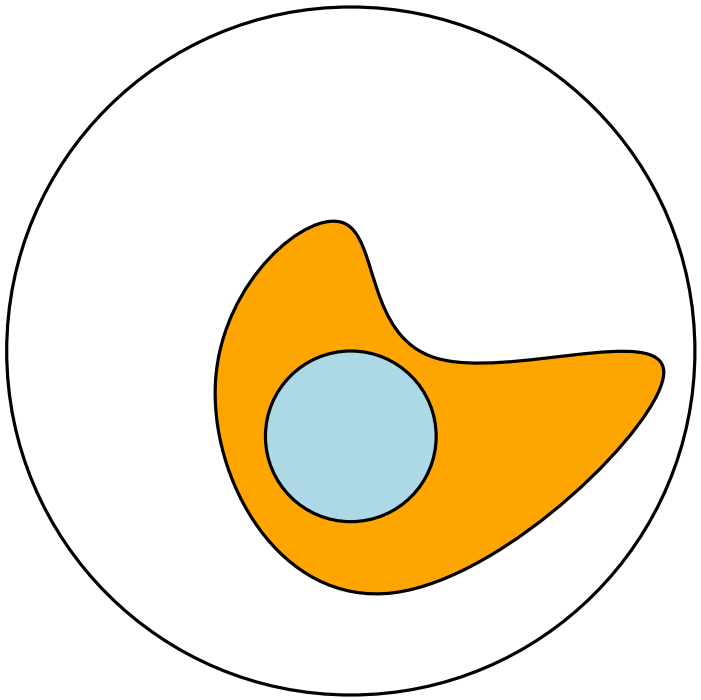

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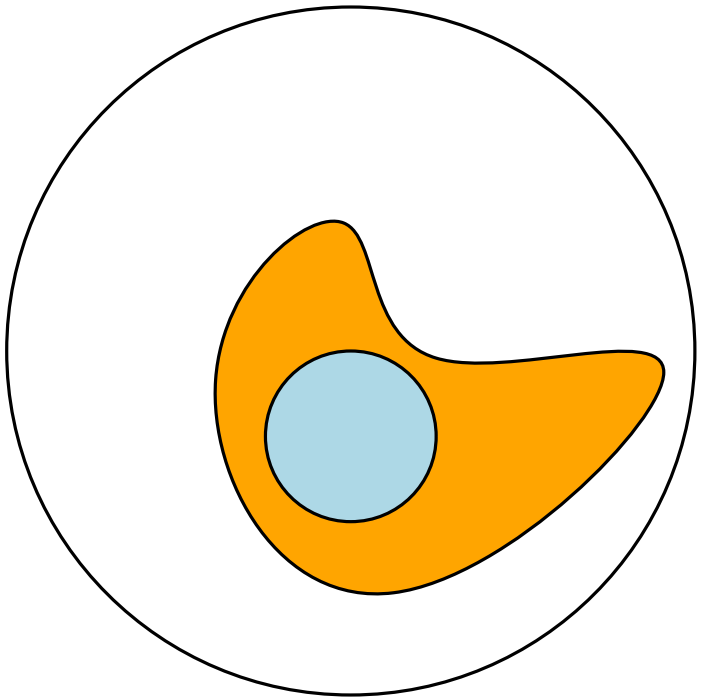

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- Use the coupling between CLE_κ and $\mathrm{CLE}_{16/\kappa}$ from CLE percolations (Miller, Sheffield, Werner '17)

Existence of geodesics: an useful exponent

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Proposition (estimate for two disjoint metric balls)
Let $\delta > 0$. Then for all $x, y, z \in \mathbb{D}$ with $|x - z| \wedge |y - z| \geq \delta$,

$$\mathbb{P}\left(\mathscr{B}^{B_\varepsilon(z)}(\mathscr{L}(x)) \cap \mathscr{B}^{B_\varepsilon(z)}(\mathscr{L}(y)) = \emptyset\right) \leq \varepsilon^{2+o(1)} \quad \text{as } \varepsilon \to 0.$$

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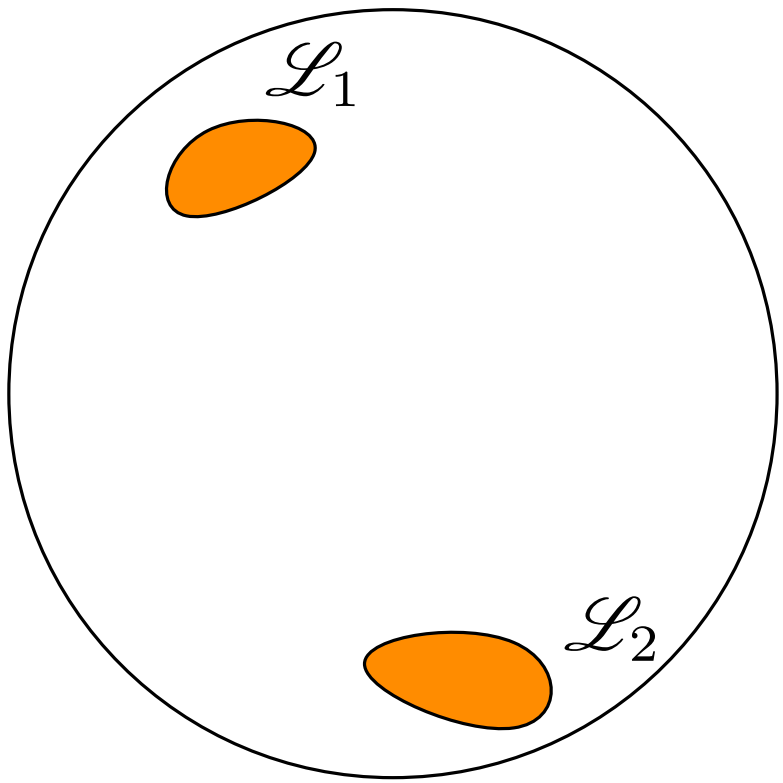
The proof uses a four-arm exponent for bi-chordal SLE_4 coming from a result of Zhan '21.

Existence of geodesics: idea of proof

Let $\mathscr{L}_1, \mathscr{L}_2 \in \Gamma$.

For all $z \in \overline{\mathbb{D}}, \mathscr{L} \in \Gamma$, let $D(\mathscr{L}, z) := \inf\{t \geq 0 : z \in \mathscr{B}_t(\mathscr{L})\}$

Let $K := \overline{\{z \in \mathbb{D} : D(\mathscr{L}_1, z) + D(\mathscr{L}_2, z) = D(\mathscr{L}_1, \mathscr{L}_2)\}}$



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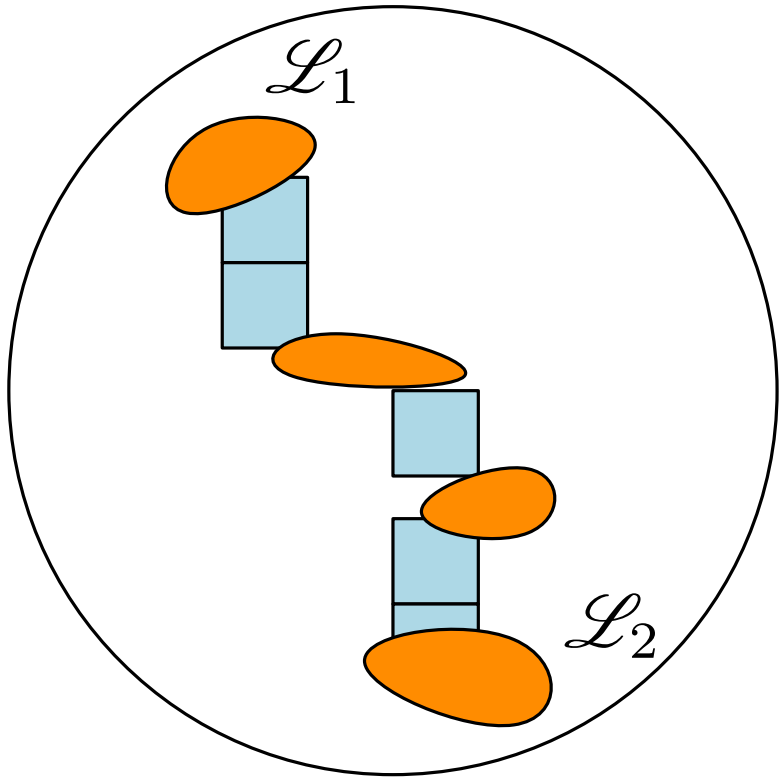
Idea: cover K by loops of size at least 2^{-n} and by dyadic squares

The previous proposition shows that there are $2^{o(n)}$ squares.



Uniqueness: covering geodesics by annuli

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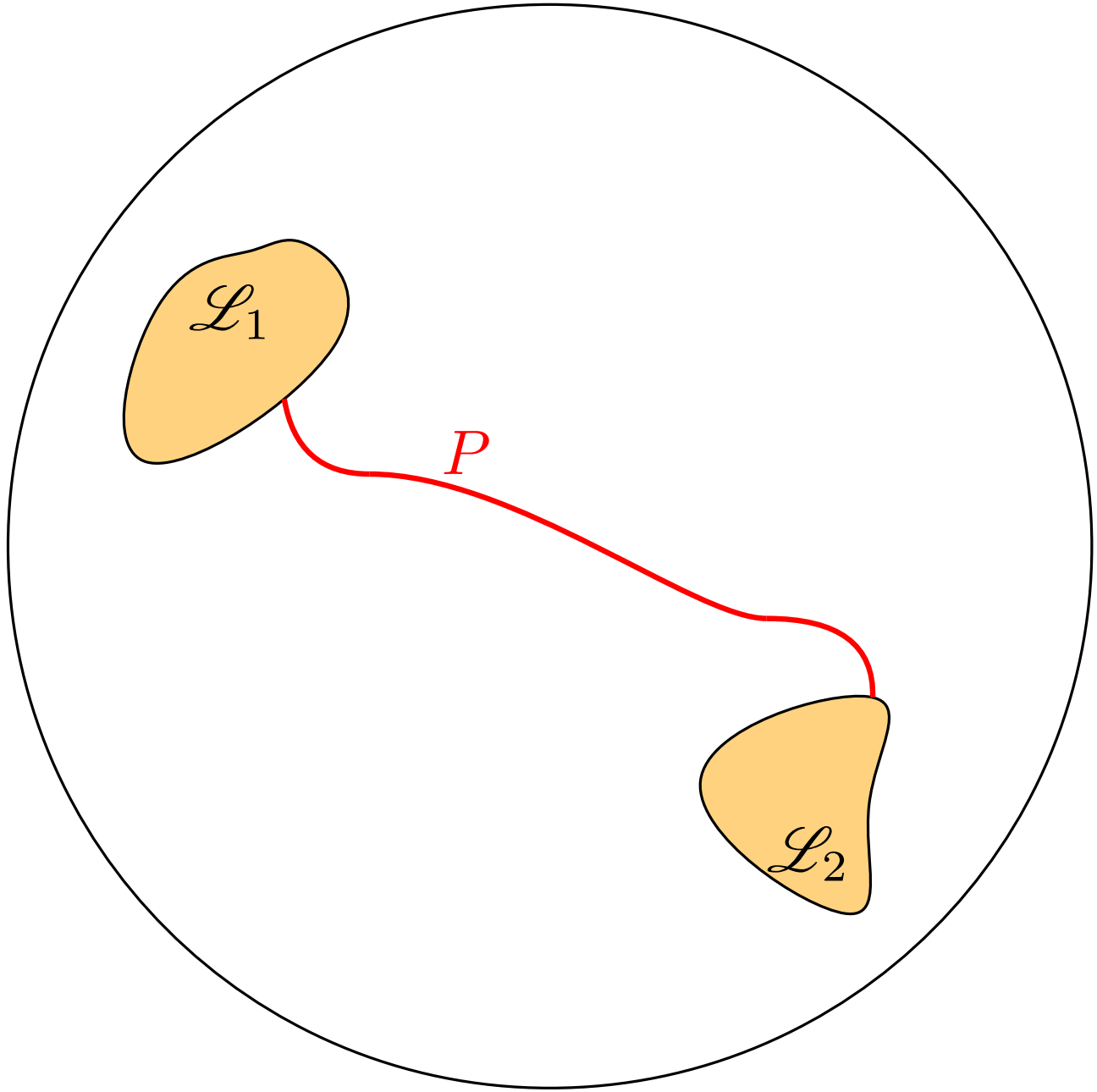

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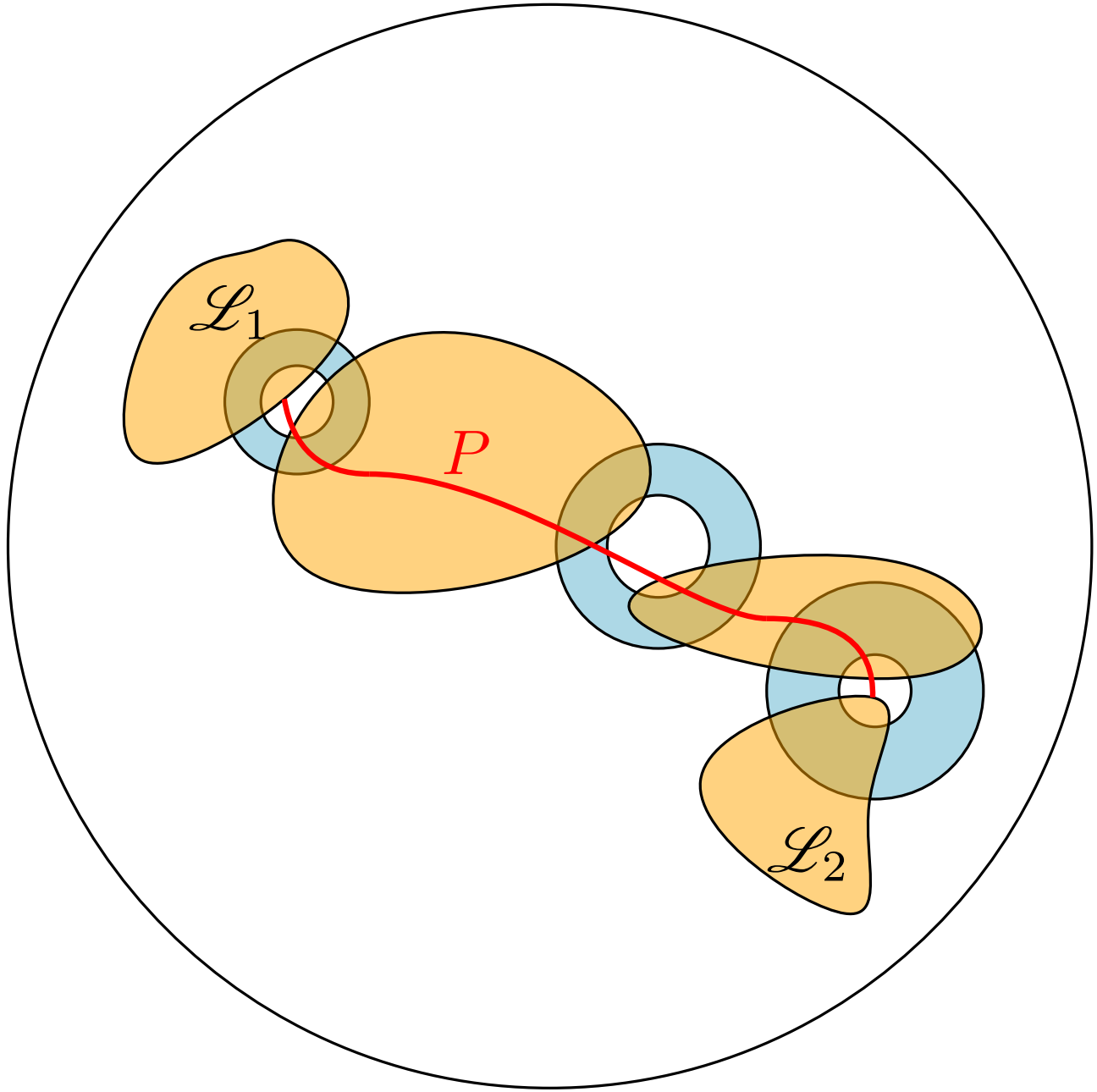

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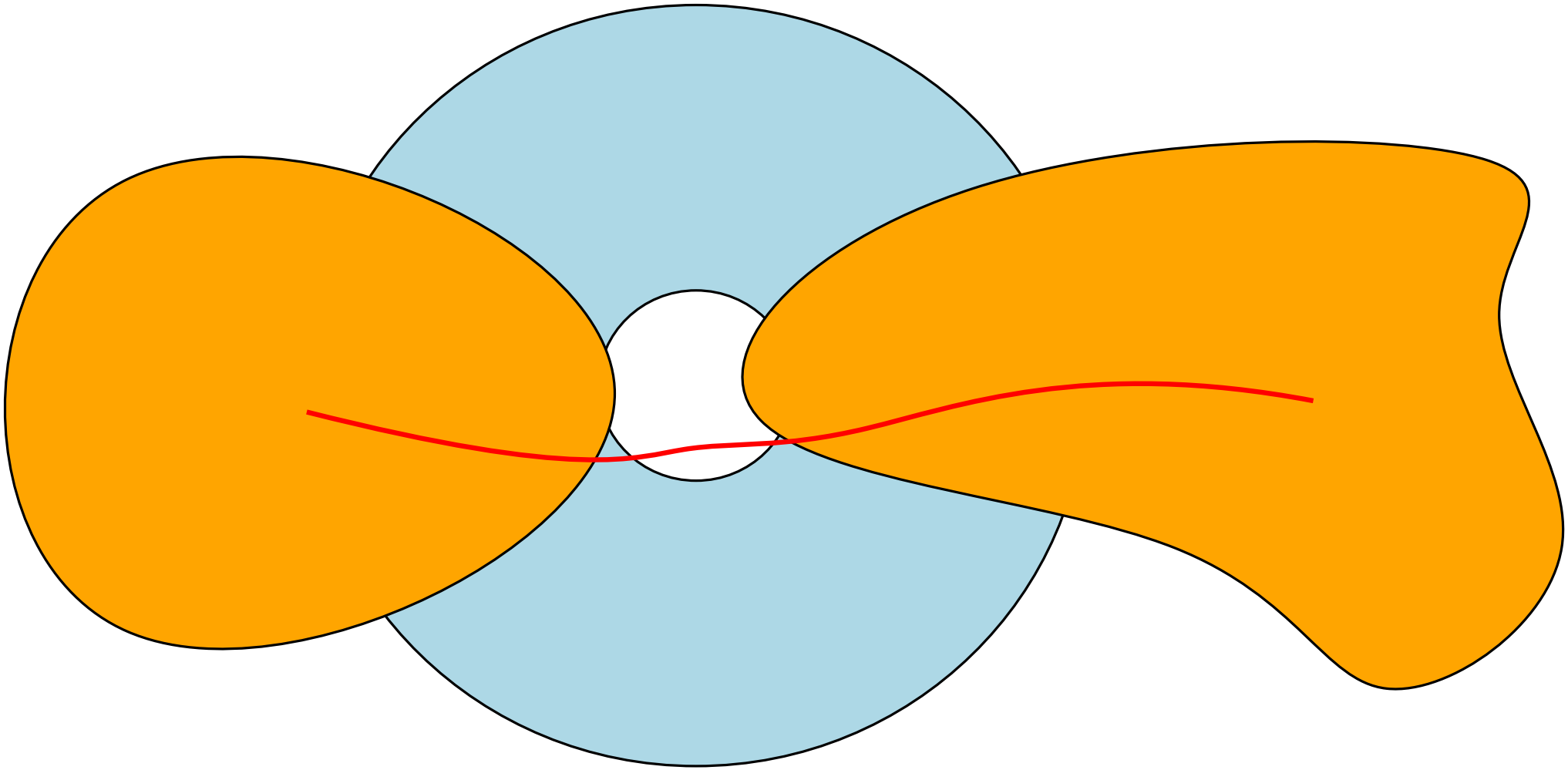

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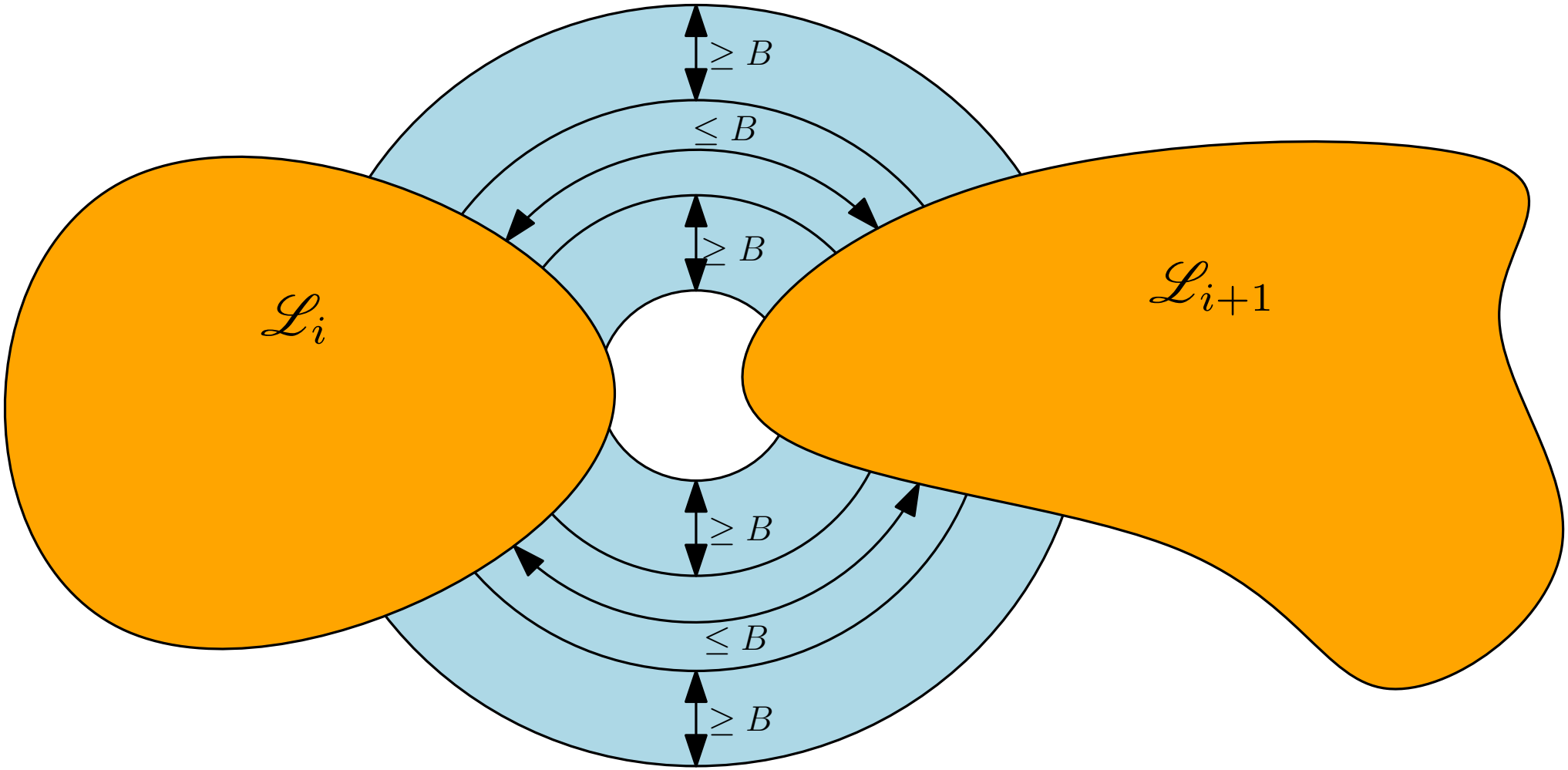

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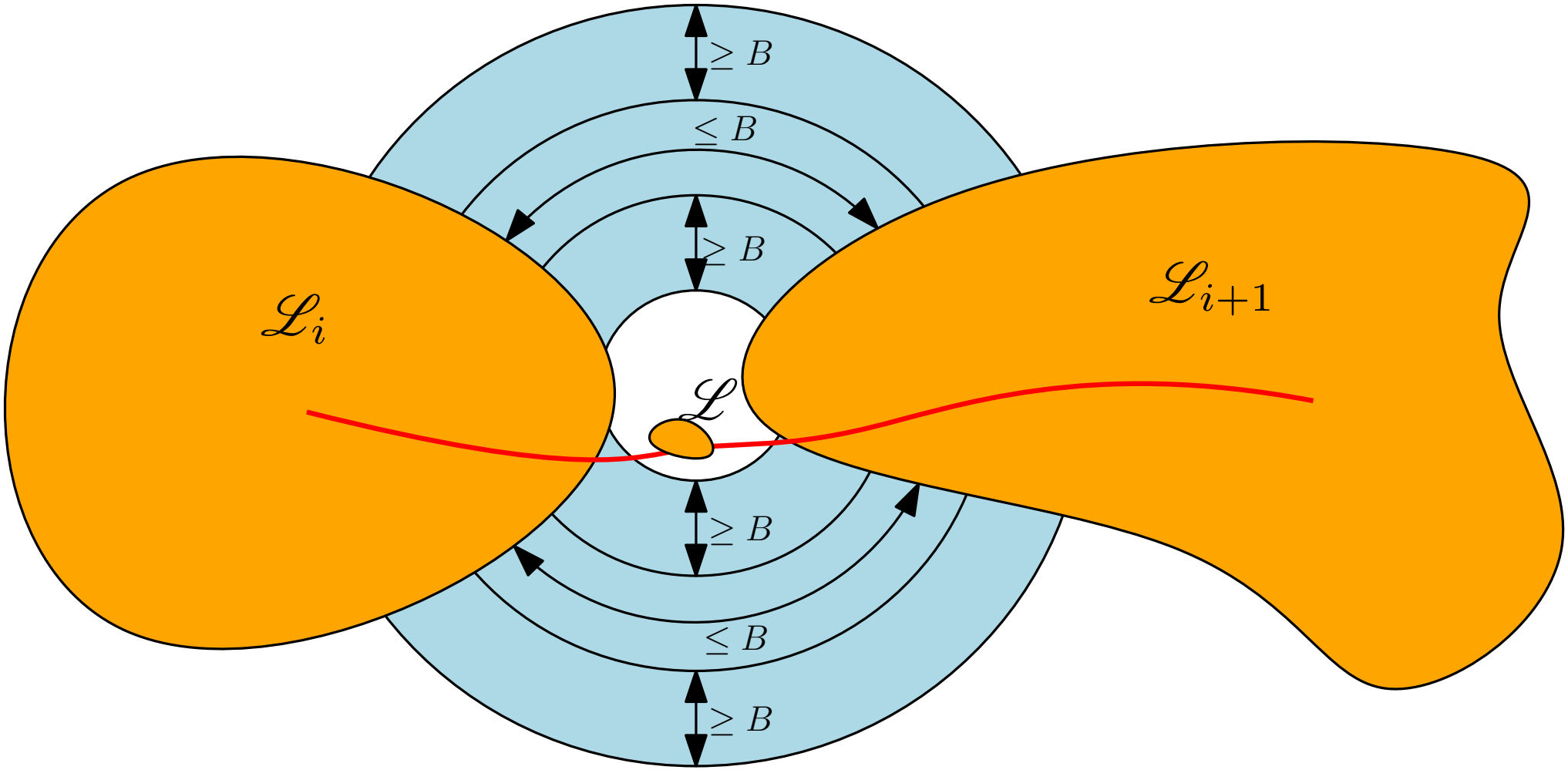

If $\mathscr{L}$ is on the geodesic, then the geodesic from $\mathscr{L}$ to $\partial\mathbb{D}$ has to hit $\mathscr{L}_i$ or $\mathscr{L}_{i+1}$.

Thanks for your attention!