Mixing of Random Laguerre Tessellation

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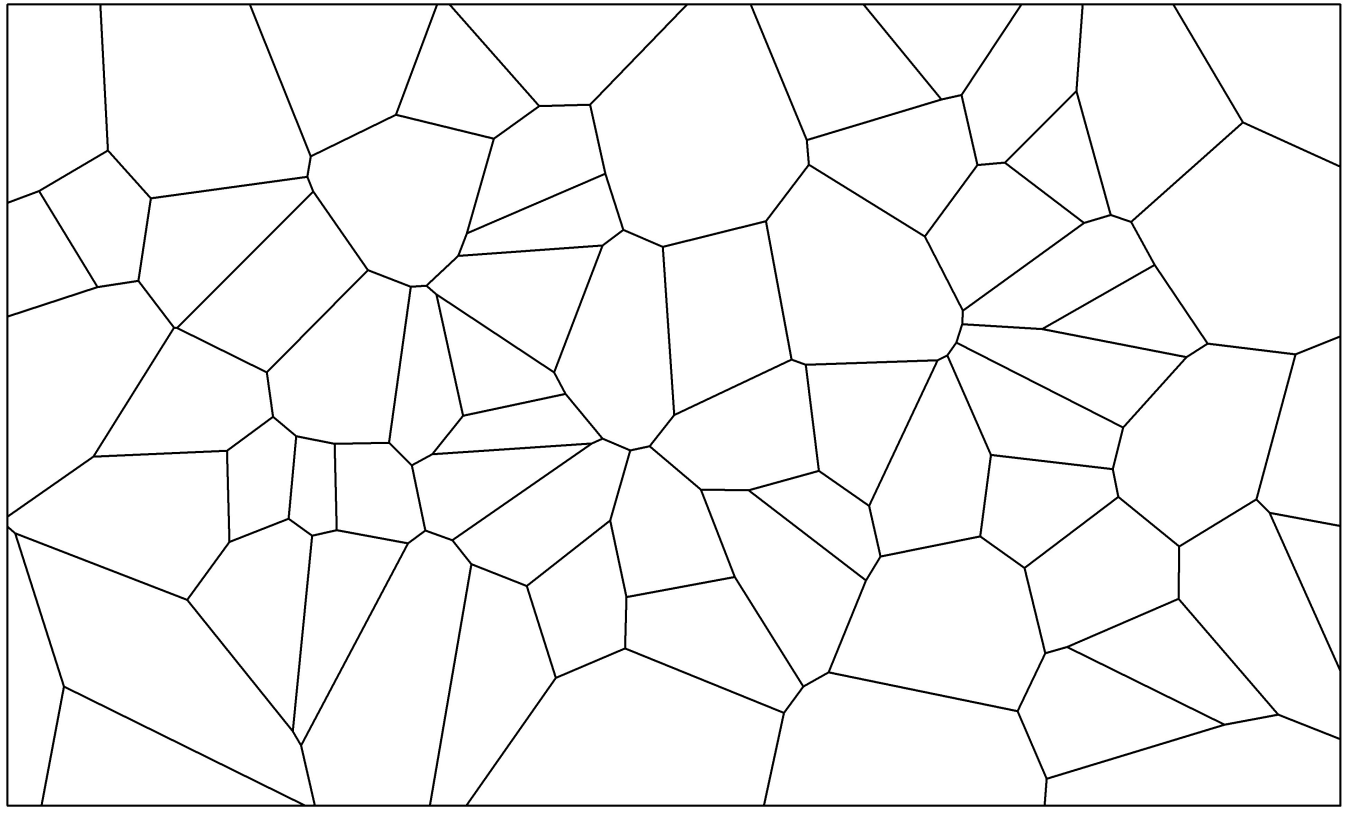
Based on joint work with Zbyněk Pawlas.

GeoSto26, Rouen, 24.9.2026

- random tessellation $\approx$ random partition of $\mathbb{R}^d$ into *nice sets*



- mixing properties $\approx$ some kind of asymptotic independence

Tessellations

- A collection of sets $\{C_i\}_{i\in\mathbb{N}}$ is called a **tessellation** of $\mathbb{R}^d$, if
  - it is locally finite,
  - it is a partition of $\mathbb{R}^d$, i. e.

$$\bigcup_{i\in\mathbb{N}} C_i = \mathbb{R}^d,$$

  - $C_i \subset \mathbb{R}^d$ is convex compact with $\operatorname{int}(C_i) \neq \emptyset$ for all $i \in \mathbb{N}$,
  - $\operatorname{int}(C_i) \cap \operatorname{int}(C_j) = \emptyset$ for $i \neq j$, $i, j \in \mathbb{N}$.

- Let $\mathcal{F}'_d := \mathcal{F}(\mathbb{R}^d)$ be all non-empty closed subsets of $\mathbb{R}^d$.
- Let $\mathbb{T}_d$ be the set of all tessellations of $\mathbb{R}^d$.

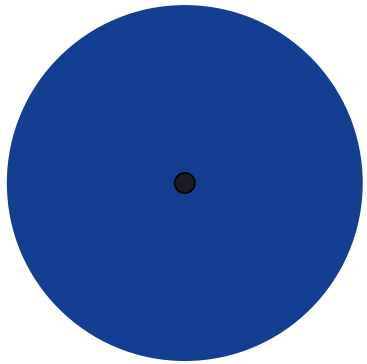
We consider $\mathbb{T}_d \subset \mathcal{F}(\mathcal{F}'_d)$ equipped with the **Fell topology**.
Random tessellation $\sim$ random element in $\mathbb{T}_d$.

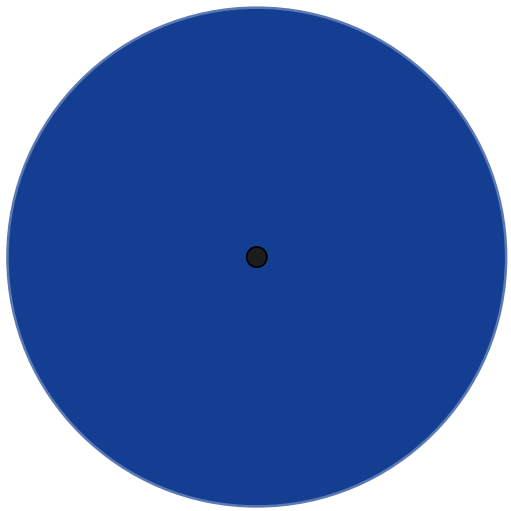
Laguerre tessellation

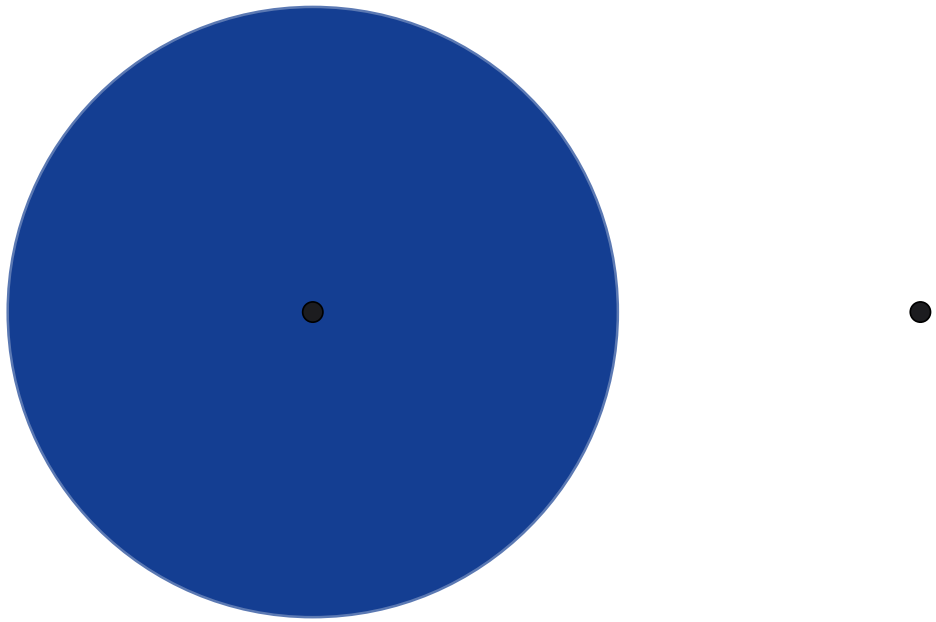
Laguerre tessellation

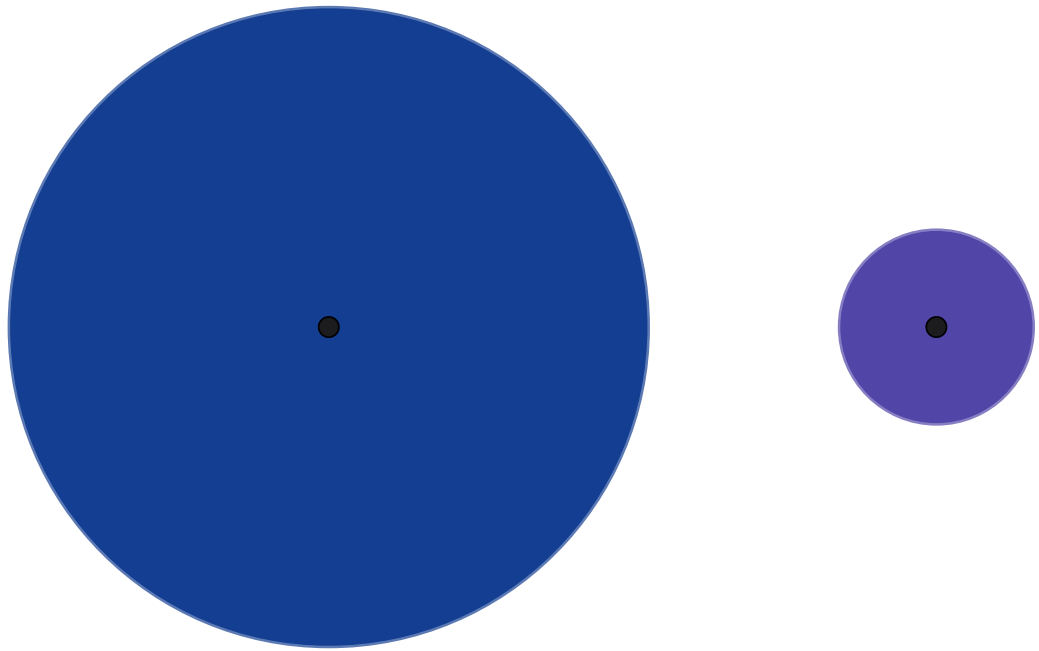
- $\varphi = \{(x_i, m_i)\}_{i \in \mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R}$ *suitable* set of marked points
- Laguerre cell of (x, m) in φ for $(x, m) \in \varphi$ is

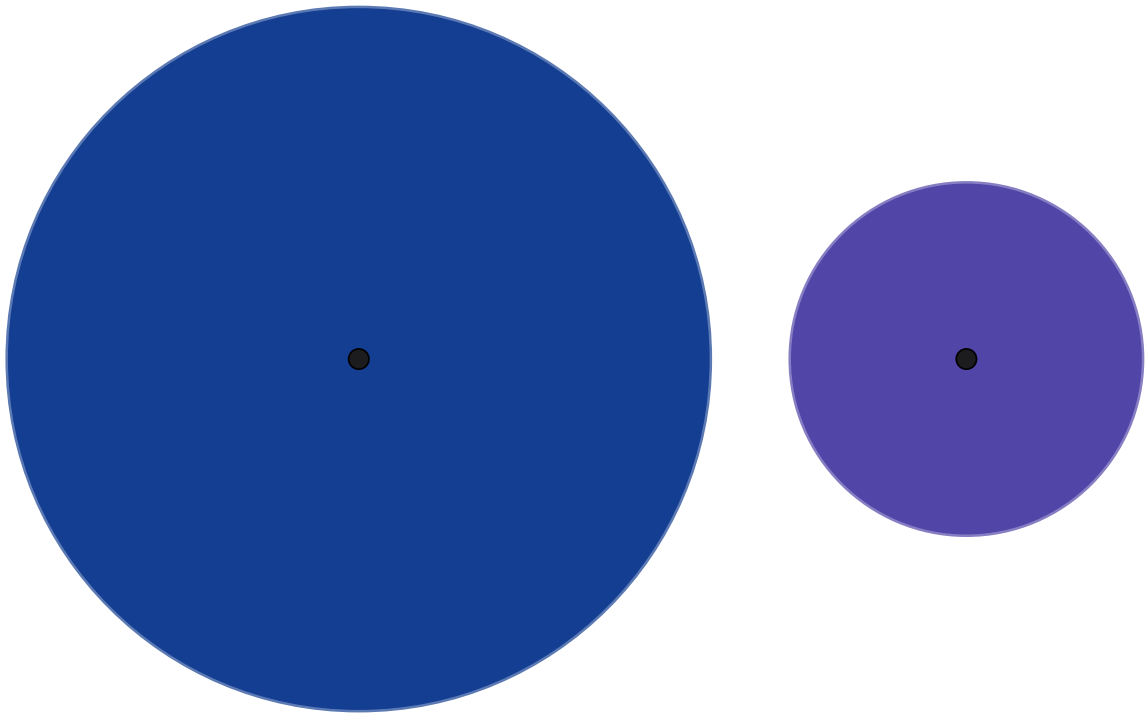
$$L((x, m), \varphi) := \{z \in \mathbb{R}^d : ||x - z||^2 + m \leq ||x_j - z||^2 + m_j \ \forall (x_j, m_j) \in \varphi\}$$

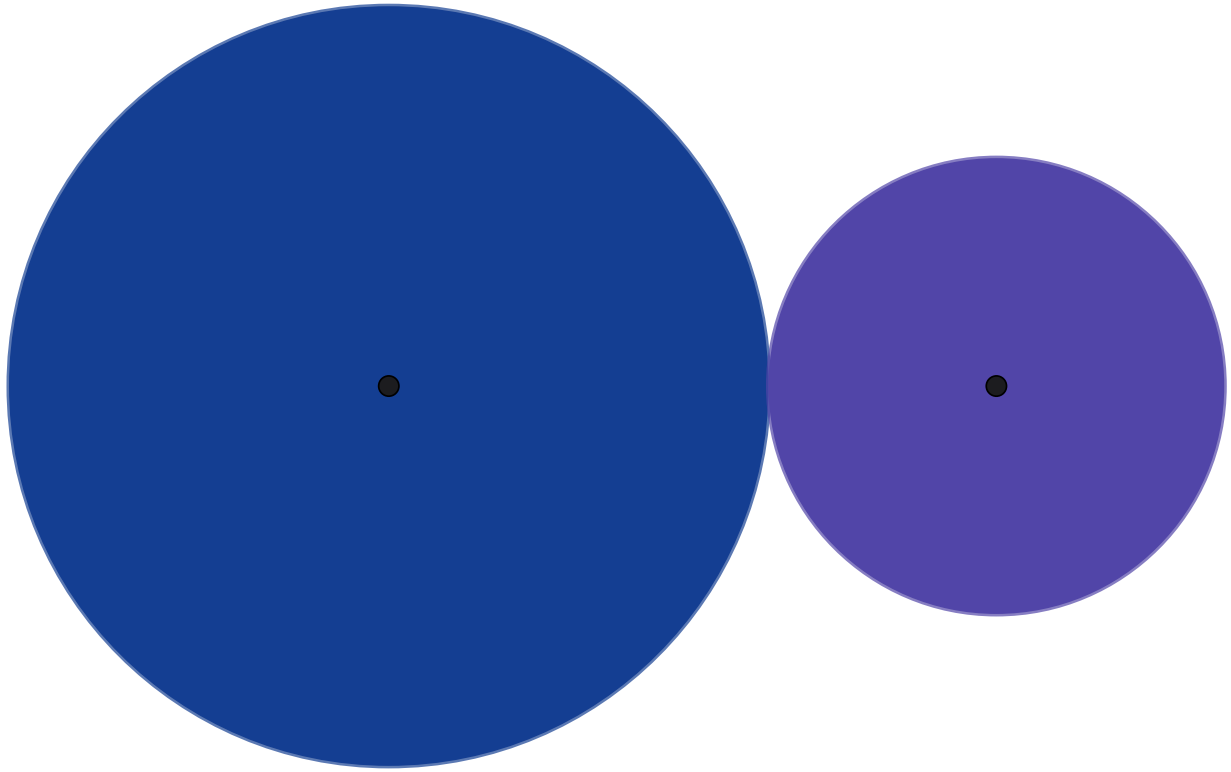


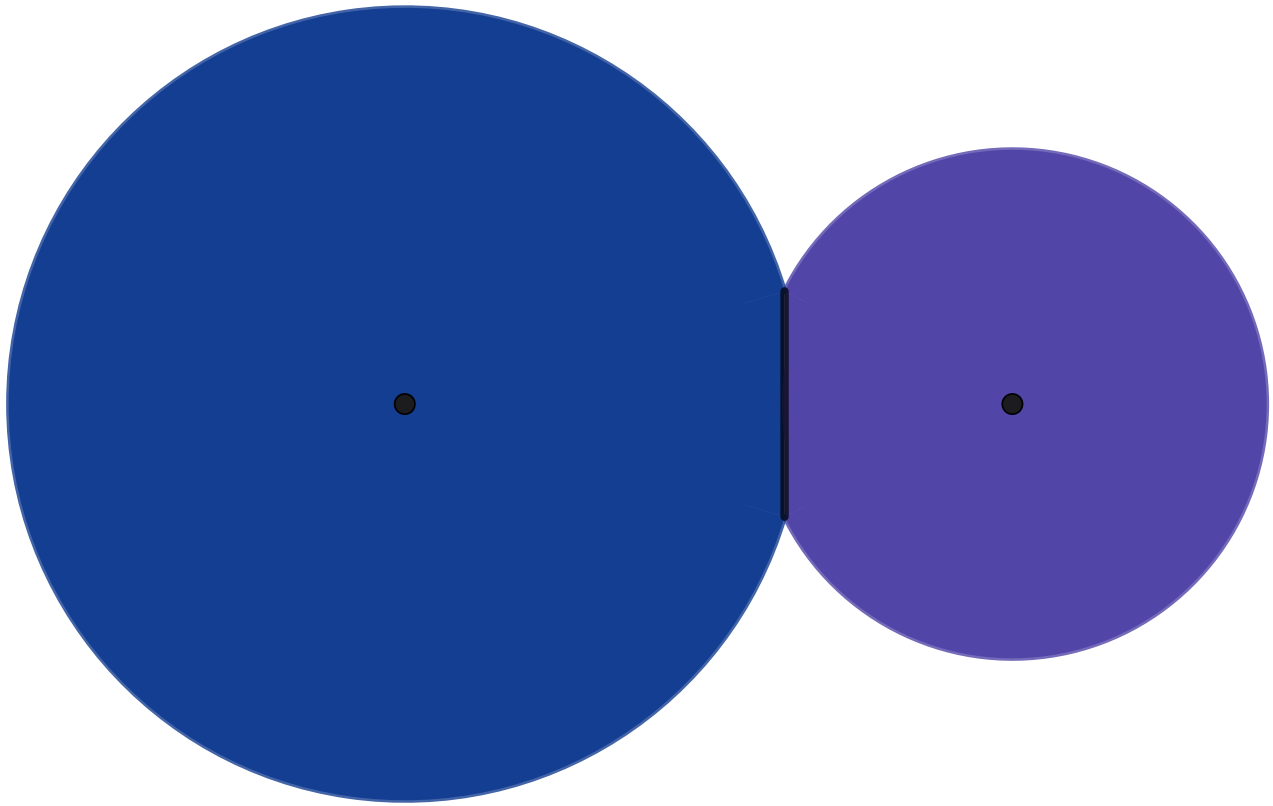


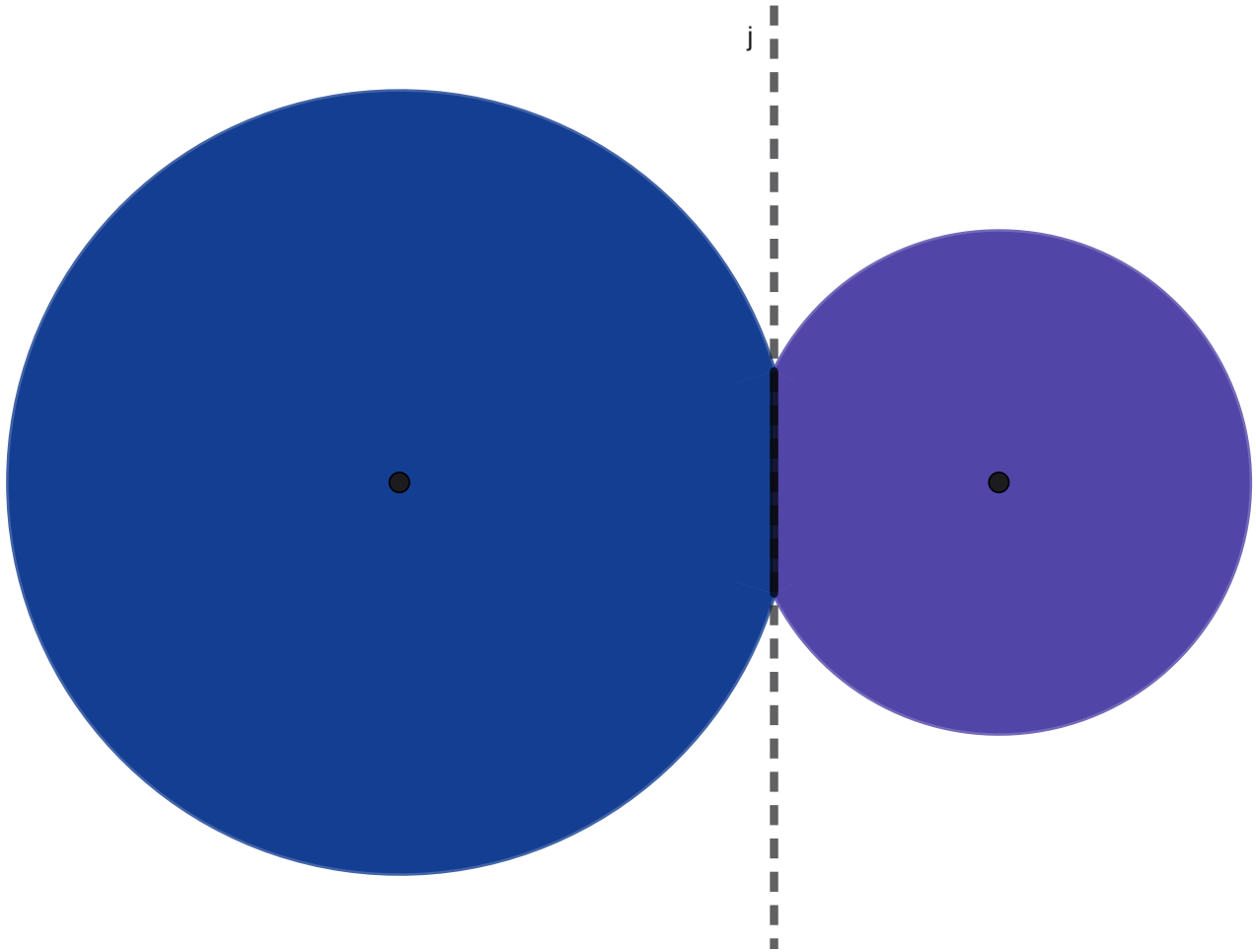








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Laguerre tessellation

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$$= \bigcap_{(x_j,m_j)\in\varphi} \{z \in \mathbb{R}^d : 2\langle z, x_j - x\rangle \leq ||x_j||^2 - ||x||^2 - m + m_j\}$$

- each cell $L((x, m), \varphi)$ is convex and compact
- Laguerre tessellation of φ is $L(\varphi) := \{L(\mathbf{x}, \varphi) : \mathbf{x} \in \varphi, \operatorname{int} L(\mathbf{x}, \varphi) \neq \emptyset\}$
- **random** Laguerre tessellation $\rightarrow$ **random** set of generators φ
  $\rightarrow$ marked point process η on $\mathbb{R}^d \times \mathbb{R}$

What is a suitable random generator η?

Theorem 4.1 in Lautensack, Zuyev (08)

Let η be an independently marked **Poisson** point process on $\mathbb{R}^d \times (-\infty, 0]$ with intensity measure $\gamma\lambda^d \otimes \mathbb{Q}$ and let $M \sim \mathbb{Q}$. Then

$$\mathbb{E}M^{d/2} < \infty \iff \mathbb{P}(L(\eta) \text{ is a tessellation}) = 1.$$

- Theorem 3.3 in Gusakova, in Wolde-Lübke (25): more general result, where η is a Poisson process on $\mathbb{R}^d \times \mathbb{R}$ with intensity measure $\lambda^d \otimes \mathbb{F}$

Key ideas:

- stationarity of $\eta' := \eta(\cdot \times \mathbb{R}) \implies \text{conv}(\eta') = \mathbb{R}^d$ a. s.
- $\mathbb{P}(\forall z \in \mathbb{R}^d, \forall t \in \mathbb{R} : \left|\{(x, m) \in \eta : ||z - x||^2 + m \leq t\}\right| < \infty) = 1$

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Using a suitable tool to control the marks, we can generalize this for larger class of marked point processes.

Definition due to Rœlly, Zass (20)

Let $\delta > 0$. We define the set of **tempered configurations** as

$$\mathcal{M}_{\text{temp},\delta} = \bigcup_{l \in \mathbb{N}} \mathcal{M}_{l,\delta},$$

where

$$\mathcal{M}_{l,\delta} = \{\varphi \subset \mathbb{R}^d \times [0,\infty) : \forall k \in \mathbb{N} \sum_{(x,m) \in \varphi \cap B(0,k) \times [0,\infty)} (1 + m^{d+\delta}) \leq l \cdot k^d\}.$$

- controls the size of the marks of point far away from the origin
- tessellations $\rightarrow$ additionally empty spaces need to be dealt with

Theorem 1(a) in Pawlas, ŠP (26+)

Let η be an ergodic stationary marked point process on $\mathbb{R}^d \times \mathbb{R}$ with a stationary mark distribution $\mathbb{Q}$. Assume that there exists $\delta > 0$ such that the typical mark $M \sim \mathbb{Q}$ satisfies

$$\mathbb{E} M_-^{\frac{d+\delta}{2}} < \infty. \tag{1}$$

Then $L(\eta)$ is a. s. a tessellation of $\mathbb{R}^d$.

- Denote

$$\hat{\eta} := \sum_{(x,m)\in\eta} \eta_{(x,\sqrt{m_-})},$$

- ergodicity and (1) $\implies$ $\mathbb{P}(\hat{\eta} \in \mathcal{M}_{\text{temp},\delta}) = 1 \implies$

$$\mathbb{P}\left(\forall z \in \mathbb{R}^d, \forall t \in \mathbb{R} : \left|\{(x,m) \in \eta : ||z-x||^2 + m \le t\}\right| < \infty\right) = 1.$$

Mixing of Random Laguerre Tessellation

Mixing properties of marked point process

Let η be a stationary marked point process on $\mathbb{R}^d \times \mathbb{R}$. We say that it is

(i) ergodic, if for all $A, B \in \mathcal{N}(\mathbb{R}^d \times \mathbb{R})$,

$$\lim_{a\to\infty} \frac{1}{\lambda^d((-a,a)^d)} \int_{(-a,a)^d} \mathbb{P}(\eta \in A, \eta + z \in B)\,\mathrm{d}z = \mathbb{P}(\eta \in A)\mathbb{P}(\eta \in B),$$

(ii) mixing, if for any $A, B \in \mathcal{N}(\mathbb{R}^d \times \mathbb{R})$,

$$\lim_{||z||\to\infty} \mathbb{P}(\eta \in A, \eta + z \in B) = \mathbb{P}(\eta \in A)\mathbb{P}(\eta \in B),$$

(iii) α-mixing, if for all $0 < a < \infty$,

$$\lim_{\Delta\to\infty} \alpha(\sigma(\eta \cap (B(0,a) \times \mathbb{R}), \sigma(\eta \cap (B(0,\Delta)^c \times \mathbb{R})) = 0,$$

where $\alpha(\mathcal{A}_1, \mathcal{A}_2) := \sup\{|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| : A_i \in \mathcal{A}_i, i = 1, 2\}$.

Mixing properties of random tessellation

Let X be a stationary random tessellation. It is called

(i) ergodic, if for all $A, B \in \mathcal{B}(\mathcal{F}(\mathcal{F}'_d))$,

$$\lim_{a\to\infty} \frac{1}{\lambda^d((-a,a)^d)} \int_{(-a,a)^d} \mathbb{P}(X \in A, X + z \in B)\,\mathrm{d}z = \mathbb{P}(X \in A)\mathbb{P}(X \in B),$$

(ii) mixing, if for any $A, B \in \mathcal{B}(\mathcal{F}(\mathcal{F}'_d))$,

$$\lim_{||z||\to\infty} \mathbb{P}(X \in A, X + z \in B) = \mathbb{P}(X \in A)\mathbb{P}(X \in B),$$

(iii) α-mixing, if for any $a > 0$,

$$\lim_{b\to\infty} \alpha(\sigma_X(\mathrm{F}^0_a), \sigma_X(\mathrm{F}^0_{-b})) = 0,$$

where

- $\sigma_X(\mathrm{F}^0_a)$ contains all information about cells intersecting $B(0, a)$
- $\sigma_X(\mathrm{F}^0_{-b})$ contains all information about cells intersecting $\mathbb{R}^d \setminus B(0, b)$

Some known results

- mixing of Poisson-hyperplane tessellation
    - D. Hug and R. Schneider (2024). *Poisson Hyperplane Tessellations.*
- mixing and β-mixing for STIT tessellations
    - R. Lachièze-Rey (2011). *Mixing properties for STIT tessellations.*
    - S. Martínez and W. Nagel (2016). *The β-mixing rate of STIT tessellations.*
- mixing and β-mixing for random (Poisson-) Voronoi
    - L. Heinrich (1994). *Normal approximation for some mean-value estimates of absolutely regular tessellations.*
    - R. Schneider and W. Weil (2008). *Stochastic and Integral Geometry.*
- β-mixing for dual of Poisson-Laguerre tessellations
    - A. Gusakova, Z. Kabluchko and C. Thäle (2023).*The β-Delaunay tessellation IV: Mixing properties and central limit theorems.*

Theorem 1 in Pawlas, ŠP (26+)

Let η be a stationary marked point process on $\mathbb{R}^d \times \mathbb{R}$ with a stationary mark distribution $\mathbb{Q}$. Assume that there exists $\delta > 0$ such that the typical mark $M \sim \mathbb{Q}$ satisfies

$$\mathbb{E} M_-^{\frac{d+\delta}{2}} < \infty.$$

Then the following implications hold.

(a) If η is ergodic, then $L(\eta)$ is a. s. a tessellation of $\mathbb{R}^d$ and it is also ergodic.

(b) If η is mixing, then $L(\eta)$ is mixing.

(c) If η is α-mixing, then $L(\eta)$ is α-mixing.

Proof Ideas

- ergodicity and mixing - easy corollary of Theorem 4 in Pawlas, ŠP(26+)

  The mapping $\varphi \to L(\varphi)$ is measurable in a suitable way.

  Then

$$\begin{aligned}
&\lim_{||z||\to\infty} \mathbb{P}(L(\eta) \in A, L(\eta) + z \in B) \\
&\qquad = \lim_{||z||\to\infty} \mathbb{P}(L(\eta) \in A, L(\eta + z) \in B) \\
&\qquad = \lim_{||z||\to\infty} \mathbb{P}(\eta \in L^{-1}(A), \eta + z \in L^{-1}(B)) \\
&\qquad = \mathbb{P}(\eta \in L^{-1}(A))\mathbb{P}(\eta \in L^{-1}(B)) = \mathbb{P}(L(\eta) \in A)\mathbb{P}(L(\eta) \in B).
\end{aligned}$$

- α-mixing - generalization of Theorem 10.5.1 in Schneider, Weil (08)

$$\begin{aligned}
&\mathbb{P}(L(\eta) \in A, L(\eta) \in B) \\
&\qquad \approx \mathbb{P}(L(\eta \cap B(0,\alpha)) \in A, L(\eta \cap \mathbb{R}^d \setminus B(0,\beta)) \in B) + C\varepsilon
\end{aligned}$$

  for A depending on cells intersecting $B(0,a)$ and B depending on cells intersecting $\mathbb{R}^d \setminus B(0,b)$.

Some open questions

- preserving of β-mixing
- preserving of speed of convergence
- general point process in $\mathbb{R}^d \times \mathbb{R}$
- necessary moment condition

Thank you for your attention.

Z. Pawlas, MŠP (2026+): Mixing Properties of Random Laguerre Tessellations, *ArXiv preprint.*

References:

A. Gusakova and M. in Wolde-Lübke (2025). Poisson–Laguerre tessellations. *Electronic Journal of Probability 30, 1–48.*

C. Lautensack, S. Zuyev (2008). Random Laguerre tessellations. *Advances in Applied Probability 40(3), 630–650.*

S. Roelly, A. Zass (2020): Marked Gibbs point processes with unbounded interaction: an existence result. *Journal of Statistical Physics, 179, 972–996.*

R. Schneider and W. Weil (2008): Stochastic and Integral Geometry. *Probability and its Applications. Springer-Verlag, Berlin.*

Acknowledgement: This work was supported by the Charles University Grant Agency (project no. 70524).